THEOREMS ON THE ALLIED SERIES OF A FOURIER SERIES

有關富氏級數之共軛級數二定理

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1. Introduction

During the last fifty years an enormous progress has been made in the development of the summability of a Fourier series and its allied series. The summability of these series has occupied the time and energies of some of the leading mathematicians of the present century.

In this paper we shall prove two new theorems concerning the allied series of a Fourier series.

We assume that f(x) is integrable in the sense of Lebesgue and is periodic with period 2π. Let the Fourier series of f(x) be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} c_n$$

so that its allied series is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} \bar{c}_n. \qquad (1)$$

We write, for $\alpha > -1$,

$$s_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{r=0}^{n} A_{n-r}^{\alpha-1} s_r = \frac{1}{A_n^{\alpha}} \sum_{r=0}^{n} A_{n-r}^{\alpha} c_r,$$

$$\bar{t}_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{r=1}^{n} A_{n-r}^{\alpha-1}\, r\, \bar{c}_r,$$

where

$$s_n = s_n^{\circ} = c_{\circ} + c_1 + \cdot \cdot \cdot \cdot + c_n, \quad A_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)},$$

The series Σc_n is said to be absolutely summable (c,α) or summable $|c,\alpha|$, where $\alpha \geq 0$, if the series $\Sigma \left| s_n^{\alpha} - s_{n-1}^{\alpha} \right|$ converges. The sequence s_n is also said to converge $|c,\alpha|$ to the limit $\lim s_n^{\alpha}$. We also write

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

and for $t > 0$,

$$\Phi_{\alpha}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} \varphi(u)\,du \qquad (\alpha > 0)$$

$$\Phi_0(t) = \varphi(t)$$

$$\varphi_{\alpha}(t) = \Gamma(\alpha+1)t^{-\alpha}\,\Phi_{\alpha}(t) \qquad (\alpha \geq 0).$$

We employ $\Psi_{\alpha}(t)$, $\psi_{\alpha}(t)$ with similar meanings.

In 1936 Bosanquet proved that* if $\varphi_{\alpha}(t)$ is of bounded variation in $(0,\pi)$, then the Fourier series of $f(t)$ is summable $|c,\beta|$ at the point $t=x$ where $\beta>\alpha\geq 0$. With regard to the sequence $n\bar{c}_n$ Chow proved that† if $\alpha \geq 0$ and $\psi_{\alpha}(t)$ is of bounded variation in $(0,\pi)$, then the sequence $n\bar{c}_n$ converges (c,β) to $2\pi^{-1}\psi_{\alpha}(+0)$ as $n \to \infty$ for $\beta > \alpha$ or $\beta \geq \alpha$ according as $0 \leq \alpha < 1$ or $\alpha \geq 1$. Bosanquet and Hyslop stated, without proof, that,‡ if $\psi(t)$ is of bounded variation in $(0,\pi)$, then the sequence $n\bar{c}_n$ converges $|c,\beta|$ to $2\pi^{-1}\psi(+0)$ for $\beta > 1$.

In this paper we shall prove the following

***Theorem A**. If $\alpha > 0$ and $\psi_{\alpha}(t)$ is of bounded variation in $(0,\pi)$, then the sequence $n\bar{c}_n$ converges $|c,\beta|$ for every $\beta > \alpha + 1$.*

Hardy and Littlewood proved that** if $1 < p \leq 2$ and $\int_0^{\pi} t^{-1}\left|\varphi(t)-s\right|^{p}dt$

* Bosanquet (1).

† Chow (4).

‡ Bosanquet and Hyslop (3).

**Hardy and Littlewood (7).

$< \infty$, then the series $\Sigma\, n^{-1} \left| \overset{\circ}{s}_n - s \right|^k$ is convergent for $k \geq p$. In 1951 Chow proved that|| under the same hypothesis the series $\Sigma\, n^{-1} \left| s_n^{\alpha} - s \right|^p$ is convergent for $\alpha = \frac{1}{p} - 1$.

We shall prove the following analogous theorem concerning the allied series.

Theorem B. *If* $1 < p \leq 2$ *and* $\int_0^{\pi} t^{-1} \left| \psi(t) \right|^p dt < \infty$, *then the series* $\sum_{n=1}^{\infty} n^{-1} \left| \bar{t}_n^{\,\alpha} \right|^p$ *is convergent, where* $\alpha = 1/p$.

2. Proof of Theorem A.

In order to simplify the proof of this theorem we first prove some results given below as lemmas.

Lemma 1. *Let* S_n^{α} *and* t_n^{α} *be the nth Cesàro means of order* $\alpha > -1$ *of the sequences* $S_n = a_0 + a_1 + \cdot \cdot \cdot + a_n$ *and* $t_n = na_n$, *respectively. Then*

$$t_n^{\alpha} = n(S_n^{\alpha} - S_{n-1}^{\alpha}),$$

$$t_n^{\alpha+1} = (\alpha + 1)(S_n^{\alpha} - S_n^{\alpha+1}).$$

This lemma is well known.*

Let $\alpha > -1$. We write

$$G_n^{\alpha} = G_n^{\alpha}(t) = \frac{1}{A_n^{\alpha}}\left(\tfrac{1}{2} A_n^{\alpha-1} + \sum_{r=1}^{n} A_{n-r}^{\alpha-1} \cos rt \right)$$

$$H_n^{\alpha} = H_n^{\alpha}(t) = G_n^{\alpha} - G_{n-1}^{\alpha}.$$

Lemma 2. *If* $\alpha > 1$ *and* k *is a positive integer or zero, then*†

|| Chow (5).

* E. Kogbetliantz (9).

† A is a constant, which is independent of n and t, but not necessarily the same one each time it occurs.

$$\left|\left(\frac{d}{dt}\right)^k H_n^{\alpha}(t)\right| \begin{cases} \leq A\, n^{k-1} & (0 \leq t \leq \pi) \quad (2) \\ \leq A\, n^{k-\alpha} t^{-\alpha+1} & (\frac{1}{n} < t \leq \pi,\ k > \alpha-2) \quad (3) \\ \leq A\, n^{-2} t^{-k-1} & (\frac{1}{n} < t \leq \pi,\ k \leq \alpha-2). \quad (4) \end{cases}$$

Proof. By *Lemma 1* we have

$$H_n^{\alpha} = G_n^{\alpha} - G_{n-1}^{\alpha} = \frac{\alpha}{n}(G_n^{\alpha-1} - G_n^{\alpha})$$

$$= \frac{\alpha}{n\,A_n^{\alpha-1}}\left[\frac{-n}{2(n+\alpha)(\alpha-1)}A_n^{\alpha-2} + \sum_{r=1}^{n} \frac{r\alpha-n}{(n+\alpha)(\alpha-1)} A_{n-r}^{\alpha-2} \cos rt\right],$$

$$\left(\frac{d}{dt}\right)^k H_n^{\alpha} = \frac{\alpha}{n\,A_n^{\alpha-1}} \sum_{r=1}^{n} \frac{r\alpha-n}{(n+\alpha)(\alpha-1)} A_n^{\alpha-2}\, r^k \cos\left(rt + \frac{k\pi}{2}\right),$$

where k is any positive integer. It follows that

$$\left|H_n^{\alpha}\right| \leq A n^{-\alpha}\left\{\left|\frac{-n}{2(n+\alpha)\,(\alpha-1)}\right| A_n^{\alpha-2} + \sum_{r=1}^{n}\left|\frac{r\alpha-n}{(n+\alpha)\,(\alpha-1)}\right| A_{n-r}^{\alpha-2}\right\}$$

$$\leq A n^{-1},$$

$$\left|\left(\frac{d}{dt}\right)^k H_n^{\alpha}\right| \leq A\, n^{-\alpha} \sum_{r=1}^{n} \left|\frac{r\alpha-n}{(n+\alpha)(\alpha-1)}\right| r^k A_{n-r}^{\alpha-2}$$

$$\leq A\, n^{-\alpha} \sum_{r=1}^{n} (n-r)^k A_r^{\alpha-2} \leq A\, n^{k-1}.$$

Hence (2) is true.

Now,

$$H_n^{\alpha} = G_n^{\alpha} - G_{n-1}^{\alpha}$$

$$= \frac{1}{A_n^{\alpha}} \sum_{r=0}^{n} A_{n-r}^{\alpha-2} \frac{\sin(r+\frac{1}{2})t}{2\sin(t/2)} - \frac{1}{A_{n-1}^{\alpha}} \sum_{r=0}^{n-1} A_{n-1-r}^{\alpha-2} \frac{\sin(r+\frac{1}{2})t}{2\sin(t/2)}$$

$$= \frac{-1}{(2\sin\frac{t}{2})A_n^{\alpha}} \operatorname{Im}\left[\sum_{r=0}^{n} A_r^{\alpha-2} e^{-i(n-r+\frac{1}{2})t}\right] + \frac{1}{(2\sin\frac{t}{2})A_{n-1}^{\alpha}} \operatorname{Im}\left[\sum_{r=0}^{n-1} A_r^{\alpha-2} e^{-i(n-r-\frac{1}{2})t}\right]$$

$$=\frac{1}{(2\sin\frac{t}{2})A_n^{\alpha}}\mathrm{Im}\Big[e^{-int}(e^{\frac{it}{2}}-e^{-\frac{it}{2}})\sum_{r=0}^{n}A_r^{\alpha-2}e^{irt}+\frac{\alpha}{n}e^{-i(n-\frac{1}{2})t}\sum_{r=0}^{n-1}A_r^{\alpha-2}e^{irt}\ A_n^{\alpha-2}e^{\frac{it}{2}}\Big]$$

$$=\frac{1}{A_n^{\alpha}}\mathrm{Im}\Big[ie^{-int}\sum_{r=0}^{n}A_r^{\alpha-2}e^{irt}\Big]+\frac{\alpha}{(2\sin\frac{t}{2})nA_n^{\alpha}}\mathrm{Im}\Big[e^{-i(n-\frac{1}{2})t}\sum_{r=0}^{n-1}A_r^{\alpha-2}e^{irt}\Big]-\frac{A_n^{\alpha-2}}{2A_n^{\alpha}}.$$

Let $\varphi(\alpha,n,t)=\sum_{p=0}^{n}A_p^{\alpha-1}e^{ipt}$. Then

$$\varphi(\alpha,n,t)=\sum_{p=0}^{n}A_{n-p}^{\alpha-1}\,e^{i(n-p)t}=\sum_{p=0}^{n}A_{n-p}^{\alpha-2}\sum_{m=0}^{p}e^{i(n-m)t}$$

$$=\sum_{p=0}^{n}A_{n-p}^{\alpha-2}\,\frac{e^{i(n-p)t}-e^{i(n+1)t}}{1-e^{it}}$$

$$=\frac{1}{1-e^{it}}\sum_{p=0}^{n}A_p^{\alpha-2}\,e^{ipt}-\frac{e^{i(n+1)t}}{1-e^{it}}A_n^{\alpha-1}$$

$$=\frac{1}{1-e^{it}}\sum_{p=0}^{n+1}A_p^{\alpha-2}\,e^{ipt}-\frac{e^{i(n+1)t}}{1-e^{it}}A_{n+1}^{\alpha-1}$$

$$=\frac{1}{1-e^{it}}\,\varphi(\alpha-1,\,n+1,\,t)-\frac{e^{i(n+1)t}}{1-e^{it}}A_{n+1}^{\alpha-1}.$$

Repeating this process, we obtain

$$\varphi(\alpha,n,t)=\frac{1}{(1-e^{it})^s}\,\varphi(\alpha-s,\,n+s,\,t)-\sum_{r=1}^{s}\frac{A_{n+r}^{\alpha-r}}{(1-e^{it})^r}\,e^{i(n+r)t}$$

where s is a positive integer.

Hence

$$H_n^{\alpha}=\frac{1}{A_n^{\alpha}}\mathrm{Im}\Big[ie^{-int}\varphi(\alpha-1,n,t)\Big]+\frac{\alpha}{(2\sin\frac{t}{2})nA_n^{\alpha}}\mathrm{Im}\Big[e^{-i(n-\frac{1}{2})t}\varphi(\alpha-1,n-1,t)\Big]-\frac{A_n^{\alpha-2}}{2A_n^{\alpha}}$$

$$=H_1+H_2+H_3,\ \text{say}.$$

We choose $s>\alpha-1$. Then

$$H_1=\frac{1}{A_n^{\alpha}}\text{Im}\Big[\frac{ie^{-int}}{(1-e^{it})^s}\varphi(\alpha-s-1,n+s,t)-i\sum_{r=1}^{s}\frac{A_{n+r}^{\alpha-r-1}e^{irt}}{(1-e^{ir})^r}\Big]$$

$$=\frac{1}{A_n^{\alpha}}\text{Im}\Big[\frac{ie^{-int}}{(1-e^{it})^{\alpha-1}}-\frac{ie^{-int}}{(1-e^{it})^s}\sum_{p=n+s+1}^{\infty}A_p^{\alpha-s-2}e^{ipt}-i\sum_{r=1}^{s}\frac{A_{n+r}^{\alpha-r-1}e^{irt}}{(1-e^{it})^r}\Big]$$

$$H_2=\frac{\alpha}{(2\sin\frac{t}{2})nA_n^{\alpha}}\text{Im}\Big[\frac{e^{-i(n-\frac{1}{2})t}}{(1-e^{it})^s}\varphi(\alpha-s-1,n+s-1,t)-\sum_{r=1}^{s}\frac{A_{n+r-1}^{\alpha-r-1}e^{i(r-\frac{1}{2})t}}{(1-e^{it})^r}\Big]$$

$$=\frac{\alpha}{(2\sin\frac{t}{2})nA_n^{\alpha}}\text{Im}\Big[\frac{ie^{-i(n-\frac{1}{2})t}}{(1-e^{it})^{\alpha-1}}-\frac{e^{-i(n-\frac{1}{2})t}}{(1-e^{it})^s}\sum_{p=n+s}^{\infty}A_p^{\alpha-s-2}e^{ipt}-\sum_{r=1}^{s}\frac{A_{n+r-1}^{\alpha-r-1}e^{i(r-\frac{1}{2})t}}{(1-e^{it})^r}\Big]$$

It is easy to deduce that, for $\frac{1}{n}<t\leq\pi$ and $k\geq 0$,

$$\Big|\left(\frac{d}{dt}\right)^k\frac{1}{(1-e^{it})^r}\Big|\leq A\,t^{-(r+k)}$$

$$\Big|\left(\frac{d}{dt}\right)^k\frac{1}{(\sin\frac{t}{2})(1-e^{it})^r}\Big|\leq At^{-(r+k+1)}.$$

Hence, for $\frac{1}{n}<t\leq\pi$, we have

$$\Big|H_1\Big|\leq A\,n^{-\alpha}\,t^{-\alpha+1}+A\,n^{-\alpha}\,t^{-s}\,n^{\alpha-s-1}+A\,n^{-\alpha}\sum_{r=1}^{s}n^{\alpha-r-1}\,t^{-r}$$

$$\leq A\,n^{-\alpha}\,t^{-\alpha+1}+An^{-1}(nt)^{-s}+A\,n^{-1}(nt)^{-1},$$

$$\Big|H_2\Big|\leq A(nt)^{-1}n^{-\alpha}t^{-\alpha+1}+A(nt)^{-1}n^{-1}(nt)^{-s}+A(nt)^{-1}n^{-1}(nt)^{-1}$$

$$\leq An^{-\alpha}t^{-\alpha+1}+An^{-1}(nt)^{-s}+An^{-1}(nt)^{-1},$$

If $\alpha<2$, then

$$\Big|H_1\Big|\leq An^{-\alpha}t^{-\alpha+1}+An^{-\alpha}t^{-\alpha+1}(nt)^{\alpha-s-1}+An^{-\alpha}t^{-\alpha+1}(nt)^{\alpha-2}$$

$$\leq An^{-\alpha}t^{-\alpha+1},$$

$$\left| H_2 \right| \leq A n^{-\alpha} t^{-\alpha+1} \text{, since } s > \alpha - 1.$$

If $\alpha \geq 2$, then

$$\left| H_1 \right| \leq A n^{-2} t^{-1} (nt)^{-\alpha+2} + A n^{-2} t^{-1} (nt)^{-s+1} + A n^{-2} t^{-1}$$

$$\leq A n^{-2} t^{-1} (nt)^{-\alpha+2} + A n^{-2} t^{-1} (nt)^{-\alpha+2} + A n^{-2} t^{-1} \leq A n^{-2} t^{-1},$$

$$\left| H_2 \right| \leq A n^{-2} t^{-1}.$$

It follows that

$$\left| H_n^{\alpha} \right| \leq \left| H_1 \right| + \left| H_2 \right| + \left| H_3 \right| \leq A n^{-\alpha} t^{-\alpha+1} + A n^{-\alpha} t^{-\alpha+1} + A n^{-2}$$

$$\leq A n^{-\alpha} t^{-\alpha+1}$$

if $\alpha < 2$; and

$$\left| H_n^{\alpha} \right| \leq \left| H_1 \right| + \left| H_2 \right| + \left| H_3 \right| \leq A n^{-2} t^{-1} + A n^{-2} t^{-1} + A n^{-2}$$

$$\leq A n^{-2} t^{-1}$$

if $\alpha \geq 2$.

Thus (3) and (4) are true when $k = 0$.

The series which is obtained by differentiating term by term the series $\Sigma A_n^{\alpha-s-2} e^{int}$ k times is absolutely convergent if $\alpha - s - 2 + k < -1$ or $s > \alpha + k - 1$. Consequently, if s and k are positive integers such that $s > \alpha + k - 1$, then

$$\left| A_n^{\alpha} \left(\frac{d}{dt}\right)^k H_1 \right| \leq \left| \left(\frac{d}{dt}\right)^k \frac{e^{-int}}{(1-e^{it})^{\alpha-1}} \right| + \left| \left(\frac{d}{dt}\right)^k \sum_{p=n+s+1}^{\infty} \frac{A_p^{\alpha-s-2} e^{i(p-n)t}}{(1-e^{it})^s} \right|$$

$$+ \left| \left(\frac{d}{dt}\right)^k \sum_{r=1}^{s} \frac{A_{n+r}^{\alpha-r-1} e^{irt}}{(1-e^{it})^r} \right| = J_1 + J_2 + J_3, \text{ say;}$$

$$\left| A_n^{\alpha} \left(\frac{d}{dt}\right)^k H_2 \right| \leq \frac{\alpha}{2n} \left| \left(\frac{d}{dt}\right)^k \frac{e^{-i(n-\frac{1}{2})t}}{(\sin\frac{t}{2})(1-e^{it})^{\alpha-1}} \right| + \frac{\alpha}{2n} \left| \left(\frac{d}{dt}\right)^k \frac{\sum_{p=n+s}^{\infty} A_p^{\alpha-s-2} e^{i(p-n+\frac{1}{2})t}}{(\sin\frac{t}{2})(1-e^{it})^s} \right|$$

$$+\frac{\alpha}{2n}\left|\left(\frac{d}{dt}\right)^k \sum_{r=1}^{s} \frac{A_{n+r-1}^{\alpha-r-1} e^{i(r-\frac{1}{2})t}}{(\sin\frac{t}{2})(1-e^{it})^r}\right| = J_4 + J_5 + J_6, \text{ say.}$$

Now, for $\frac{1}{n} < t \leq \pi$

$$J_1 = \left|\sum_{m=0}^{k} \binom{k}{m} \left(\frac{d}{dt}\right)^{k-m} \frac{1}{(1-e^{it})^{\alpha-1}} \left(\frac{d}{dt}\right)^m e^{-int}\right|$$

$$\leq A \sum_{m=0}^{k} t^{-(k-m+\alpha-1)} n^m \leq A t^{-(k+\alpha-1)} \sum_{m=0}^{k} (nt)^m$$

$$\leq A\, t^{-(k+\alpha-1)} (nt)^k \quad \leq A n^k t^{-\alpha+1}.$$

And if $k \leq \alpha - 2$, then

$$J_1 \leq A n^{\alpha-2} t^{-k-1} (nt)^{k-\alpha+2} \leq A n^{\alpha-2} t^{-k-1}.$$

Similarly, we can deduce that, for $m = 2, 3, 4, 5, 6$,

$$J_m \leq A n^k t^{-\alpha+1} \quad \left(\frac{1}{n} < t \leq \pi,\ k > \alpha - 2\right),$$

$$J_m \leq A n^{\alpha-2} t^{-k-1} \quad \left(\frac{1}{n} < t \leq \pi,\ k \leq \alpha - 2\right).$$

Since

$$\left|\left(\frac{d}{dt}\right)^k H_n^{\alpha}\right| \leq \left|\left(\frac{d}{dt}\right)^k H_1\right| + \left|\left(\frac{d}{dt}\right)^k H_2\right|$$

(3) and (4) are ture for any positive integer k. The proof is thus complete.

Lemmn 3. *Let $\alpha > 0$. If $\Psi_\alpha(t)$ is of bounded variation in $(0,\eta)$, where $\eta > 0$, and $\Psi_\alpha(+0) = 0$, then for almost all t in $(0,\eta)$ and for every $\beta > \alpha - 1$, we have*

$$\Psi_\beta(t) = \frac{1}{\Gamma(1+\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha}\, d\Psi_\alpha(u).$$

This lemma is also known.*

Lemma 4. *Let $0 < \alpha < \beta - 1 < h + 1$, where $h = [\alpha]$, and let*

$$J(n,u) = \frac{1}{\Gamma(1+h-\alpha)} \int_u^\pi (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} H_n^{\beta}(t)dt.$$

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*L. S. Bosanquet (2).

Then, for $0 < u < \pi$,

$$\left| J(n,u) \right| \begin{cases} \leq A n^{\alpha-1} \\ \leq A n^{\alpha-\beta} u^{-\beta+1}. \end{cases}$$

Proof. For $u + n^{-1} < \pi$,

$$J(n,u) = \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\pi} = K_1 + K_2, \text{ say.}$$

Since

$$\left(\frac{d}{dt}\right)^{h+1} H_n^{\beta}(t) = O(n^h) \min\{1, (nt)^{-\beta+1}\}$$

we have

$$\left| K_1 \right| \leq A n^h \int_u^{u+n^{-1}} (t-u)^{h-\alpha} \min\{1, (nt)^{-\beta+1}\} dt$$

$$\leq A n^h \min\{1, (nu)^{-\beta+1}\} \int_u^{u+n^{-1}} (t-u)^{h-\alpha}\, dt$$

$$\leq A n^h \min\{1, (nu)^{-\beta+1}\}\, O(n^{-h+\alpha-1})$$

$$= O(n^{\alpha-1}) \min\{1, (un)^{-\beta+1}\},$$

while

$$K_2 = A n^{\alpha-h} \int_{u+n^{-1}}^{\xi} \left(\frac{d}{dt}\right)^{h+1} H_n^{\beta}(t) dt \qquad (u+n^{-1} < \xi < \pi)$$

$$= O(n^{\alpha-h})\, O(n^{h-1}) \min\{1, (nu)^{-\beta+1}\}$$

$$= O(n^{\alpha-1}) \min\{1, (nu)^{-\beta+1}\}.$$

Hence the lemma follows.

Lemma 5. *Let* $0 < \alpha < \beta - 1 < h + 1$, *where* $h = [\alpha]$, *and let*

$$V(n,u) = \frac{1}{\Gamma(1+\alpha)} \int_0^u v^{\alpha} \frac{d}{dt} J(n,v) dv.$$

If $\psi_{\alpha}(t)$ *is of bounded variation in* $(0,\pi)$, *then*

$$-\frac{\pi}{2}(\bar{t}_n^{\,\beta} - \bar{t}_{n-1}^{\,\beta}) = O(n^{\alpha-\beta}) + (-1)^{h+1} \psi_{\alpha}(\pi) V(n,\pi) + (-1)^h \int_0^{\pi} V(n,u) d\psi_{\alpha}(u).$$

Proof. Since

$$n\bar{c}_n=\frac{2n}{\pi}\int_0^{\pi}\psi(t)\sin nt\,dt=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\left(\frac{d}{dt}\right)\cos nt\,dt$$

we have

$$\bar{t}_n^{\beta}=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}G_n^{\beta}(t)dt,$$

$$\bar{t}_n^{\beta}-\bar{t}_{n-1}^{\beta}=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}H_n^{\beta}(t)dt.$$

Now, on integration by parts,

$$\int_0^{\pi}\psi(t)\frac{d}{dt}H_n^{\beta}(t)dt=\left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Psi_{\rho}(t)\left(\frac{d}{dt}\right)^{\rho}H_n^{\beta}(t)\right]_0^{\pi}+(-1)^h\int_0^{\pi}\Psi_h(t)\left(\frac{d}{dt}\right)^{h+1}H_n^{\beta}(t)dt.$$

Also,

$$\int_0^{\pi}\Psi_h(t)\left(\frac{d}{dt}\right)^{h+1}H_n^{\beta}(t)dt=\frac{1}{\Gamma(1+h-\alpha)}\int_0^{\pi}\left(\frac{d}{dt}\right)^{h+1}H_n^{\beta}(t)dt\int_0^{t}(t-u)^{h-\alpha}d\Psi_{\alpha}(u)$$

$$=\frac{1}{\Gamma(1+h-\alpha)}\int_0^{\pi}d\Psi_{\alpha}(u)\int_u^{\pi}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}H_n^{\beta}(t)dt$$

$$=\int_0^{\pi}J(n,u)d\Psi_{\alpha}(u)$$

$$=\Big[\Psi_{\alpha}(u)J(n,u)\Big]_0^{\pi}-\int_0^{\pi}\Psi_{\alpha}(u)\frac{d}{du}J(n,u)du.$$

Further, since $\psi_{\alpha}(+0)$ is finite,

$$\int_0^{\pi}\Psi_{\alpha}(u)\frac{d}{du}J(n,u)du=\frac{1}{\Gamma(\alpha+1)}\int_0^{\pi}\psi_{\alpha}(u)u^{\alpha}\frac{d}{du}J(n,u)du$$

$$=\frac{1}{\Gamma(\alpha+1)}\left[\psi_{\alpha}(u)\int_0^{u}v^{\alpha}\frac{d}{dv}J(n,v)dv\right]_0^{\pi}-\frac{1}{\Gamma(\alpha+1)}\int_0^{\pi}d\psi_{\alpha}(n)\int_0^{u}v^{\alpha}\frac{d}{dv}J(n,v)dv$$

$$=\psi_{\alpha}(\pi)V(n,\pi)-\int_0^{\pi}V(n,u)d\psi_{\alpha}(u).$$

Collecting our results, we have

$$-\frac{\pi}{2}(\bar{t}_n^{\ \beta}-\bar{t}_{n-1}^{\ \beta})=\Big[\sum_{\rho=1}^{h}(-1)^{\rho-1}\,\Psi_\rho(t)(\frac{d}{dt})^\rho H_n^{\ \beta}\ (t)\Big]_0^\pi+(-1)^h\Big[\Psi_\alpha(u)J(n,u)\Big]_0^\pi$$

$$+(-1)^{h+1}\,\psi_\alpha(\pi)V(n,\pi)+(-1)^h\int_0^\pi V(n,u)d\psi_\alpha(u). \qquad (5)$$

Now

$$\Big|\,(\frac{d}{dt})^\rho H_n^{\ \beta}(t)\,\Big|\le A\,n^{-2}t^{-\rho-1} \qquad (\rho\le h\text{-}1) \qquad (6)$$

$$\Big|\,(\frac{d}{dt})^h H_n^{\ \beta}(t)\,\Big|\le A\,n^{h-\beta}t^{-\beta+1}. \qquad (7)$$

Substituting (6) and (7) in (5) and applying Lemma 4, we obtain

$$-\frac{\pi}{2}(\bar{t}_n^\beta-\bar{t}_{n-1}^\beta)=\sum_{\rho=1}^{h-1}O(n^{-2})+O(n^{h-\beta})+O(n^{\alpha-\beta})$$

$$+(-1)^{h+1}\,\psi_\alpha(\pi)V(n,\pi)+(-1)^h\int_0^\pi V(n,u)d\psi_\alpha(u)$$

$$=O(n^{\alpha-\beta})+(-1)^{h+1}\,\psi_\alpha(\pi)V(n,\pi)+(-1)^h\int_0^\pi V(n,u)d\psi_\alpha(u).$$

Lemma 6. *Let* $0<\alpha<\beta\text{-}1<h+1$, *where* $h=[\alpha]$. ***Then,*** *for* $0<u<\pi$,

$$|V\,(n,u)|\ \begin{cases}\le\ A n^{\alpha-1}u^\alpha & (8)\\ \le\ A n^{\alpha-\beta}u^{\alpha-\beta+1} & (9)\end{cases}$$

and

$$|V\,(n,\pi)|\quad\le A\,n^{\alpha-\beta}. \qquad (10)$$

Proof. Let $\psi(t)=\frac{1}{2}(\pi-t)$, so that $\psi_\alpha(t)=\frac{1}{2}(\pi-\frac{t}{\alpha+1})$, and $\bar{t}_n^{\ \beta}=1$ for every positive integer n.

By *Lemma* 5 we have

$$0=-\frac{\pi}{2}(\,\bar{t}_n^{\ \beta}-\bar{t}_{n-1}^{\ \beta})$$

$$=O(n^{\alpha-\beta})+(-1)^{h+1}\psi_\alpha(\pi)V(n,\pi)+(-1)^h\int_0^\pi V(n,u)d\psi_\alpha(u)$$

$$=O(n^{\alpha-\beta})+(-1)^{h+1}\psi_{\alpha}(\pi)V(n,\pi)+\frac{(-1)^{h+1}}{2(\alpha+1)}\int_0^{\pi}V(n,u)du. \qquad (11)$$

Now,

$$\Gamma(\alpha+1)V(n,u)=u^{\alpha}J(n,u)-\alpha\int_0^{u}v^{\alpha-1}\ J(n,v)dv$$

$$=u^{\alpha}O(n^{\alpha-1})+\int_0^{u}v^{\alpha-1}\ O(n^{\alpha-1})dv$$

$$=u^{\alpha}O(n^{\alpha-1}), \qquad (12)$$

which is (8).

On the other hand,

$$\Gamma(\alpha+1)[V(n,\pi)-V(n,u)]=[v^{\alpha}J(n,v)]_u^{\pi}-\alpha\int_u^{\pi}v^{\alpha-1}\ J(n,v)dv$$

$$=O(n^{\alpha-\beta}u^{\alpha-\beta+1})+\int_u^{\pi}v^{\alpha-1}O(n^{\alpha-\beta}v^{-\beta+1})dv$$

$$=O(n^{\alpha-\beta}u^{\alpha-\beta+1}). \qquad (13)$$

From (12) and (13) we have

$$\int_0^{\pi}V(n,u)du=\int_0^{1/n}+\int_{1/n}^{\pi}$$

$$=\int_0^{1/n}O(n^{\alpha-1}u^{\alpha})du+\int_{1/n}^{\pi}\{V(n,\pi)+O(n^{\alpha-\beta}u^{\alpha-\beta+1})\}du$$

$$=O(n^{-2})+(\pi-\frac{1}{n})V(n,\pi)+O(n^{\alpha-\beta})+O(n^{-2})$$

$$=O(n^{\alpha-\beta})+(\pi-\frac{1}{n})V(n,\pi). \qquad (14)$$

It follows from (11) and (14) that

$$V(n,\pi)=O(n^{\alpha-\beta}),$$

which is (10).

By (13), we obtain

$$O(n^{\alpha-\beta})-V(n,u)=O(n^{\alpha-\beta}u^{\alpha-\beta+1})$$

from which (9) follows, since $\beta > \alpha + 1$. This completes the proof of the lemma.

Lemma 7.* *If* $\alpha \geq 0$ *and the sequence* s_n *converges* $|c,\alpha|$, *then it converges* $|c,\beta|$, *for* $\beta > \alpha$.

Proof of ***Theorem A.*** It is sufficient to prove that

$$\sum_{n=1}^{\infty} \left| \bar{t}_n^{\beta} - \bar{t}_{n-1}^{\beta} \right| < \infty.$$

We may suppose, without loss of generality, that $0 < \alpha < \beta-1 < h+1$, where $h = [\alpha]$.

Now

$$-\frac{\pi}{2}(\bar{t}_n^{\beta} - \bar{t}_{n-1}^{\beta}) = O(n^{\alpha-\beta}) + (-1)^{h+1} \psi_\alpha(\pi) V(n,\pi) + (-1)^h \int_0^{\pi} V(n,u) d\psi_\alpha(u)$$

$$= O(n^{\alpha-\beta}) + (-1)^h \int_0^{\pi} V(n,u) d\psi_\alpha(u).$$

And

$$\Sigma |V(n,u)| = \sum_{n \leq u^{-1}} O(n^{\alpha-1} u^{\alpha}) + \sum_{n > u^{-1}} O(n^{\alpha-\beta} u^{\alpha-\beta+1}) = O(1) + O(1)$$

uniformly in u, since $\beta > \alpha + 1$.

Therefore

$$\Sigma \left| \bar{t}_n^{\beta} - \bar{t}_{n-1}^{\beta} \right| \leq A \Sigma n^{\alpha-\beta} + A \Sigma \int_0^{\pi} \left| V(n,u) \right| \left| d\psi_\alpha(u) \right|$$

$$\leq A + A \int_0^{\pi} \left| d\psi_\alpha(u) \right| \Sigma \left| V(n,u) \right| \leq A + A \int_0^{\pi} \left| d\psi_\alpha(u) \right|.$$

Since $\int_0^{\pi} \left| d\psi_\alpha(u) \right|$ is finite, it follows that $\Sigma \left| \bar{t}_n^{\beta} - \bar{t}_{n=1}^{\beta} \right|$ converges.

*E. Kogbetliantz (9).

3. Proof of Theorem B.

In proving this theorem we need the following lemmas.

Lemma 8. *Let* $0<\alpha<1$ *and*

$$L_n^{\alpha}(t)=\frac{1}{A_n^{\alpha}}\sum_{r=0}^{n}A_{n-r}^{\alpha-1}\cos rt.$$

Then, for $0<t\leq\pi$,

$$L_n^{\alpha}(t)=l_n^{\alpha}(t)+k_n^{\alpha}(t)$$

where

$$l_n^{\alpha}(t)=\frac{\cos[(n+\alpha/2)t-\alpha\pi/2]}{A_n^{\alpha}(2\sin t/2)^{\alpha}},$$

$$\left|\frac{d}{dt}L_n^{\alpha}(t)\right|=O(n)\qquad(0<t\leq\pi/n),$$

$$\left|\frac{d}{dt}k_n^{\alpha}(t)\right|=O(n^{-1}t^{-2})\qquad(t>\pi/n).$$

Proof. Since

$$A_n^{\alpha}L_n^{\alpha}=-2\sin\frac{t}{2}\sum_{r=1}^{n}A_{n-r}^{\alpha}\sin(r-\tfrac{1}{2})t+A_n^{\alpha}$$

$$=-2\sin\frac{t}{2}\sum_{r=1}^{n}A_r^{\alpha}\sin(n-r-\tfrac{1}{2})t+A_n^{\alpha},$$

it is evident that, for $0<t\leq\pi/n$,

$$\left|\frac{d}{dt}L_n^{\alpha}(t)\right|=O(n).$$

Also

$$A_n^{\alpha}L_n^{\alpha}=\mathrm{Re}\left\{\sum_{r=0}^{n}A_r^{\alpha-1}e^{-i(r-n)t}\right\}$$

$$=\mathrm{Re}\left\{\frac{e^{int}}{(1-e^{-it})^{\alpha}}-\sum_{r=n+1}^{\infty}A_r^{\alpha-1}e^{-i(r-n)t}\right\}$$

$$=A_n^{\alpha}\left\{l_n^{\alpha}(t)+k_n^{\alpha}(t)\right\}$$

where

$$l_n^{\alpha}(t)=\frac{1}{A_n^{\alpha}}\ \mathrm{Re}\left\{\frac{e^{int}}{(1-e^{-it})^{\alpha}}\right\}=\frac{\cos[(n+\alpha/2)t-\alpha\pi/2]}{A_n^{\alpha}(2\sin t/2)^{\alpha}}$$

and

$$A_n^{\alpha}k_n^{\alpha}(t)=\mathrm{Re}\left\{-\frac{A_n^{\alpha-1}e^{-it}}{1-e^{-it}}-\frac{A_n^{\alpha-2}e^{-it}}{(1-e^{-it})^2}-\frac{1}{(1-e^{-it})^2}\sum_{r=n+1}^{\infty}A_r^{\alpha-3}e^{-i(r-n)t}\right\}.$$

We can easily deduce that, for $t>\pi/n$,

$$\left|\frac{d}{dt}k_n^{\alpha}(t)\right|=O(n^{-1}t^{-2}).$$

Lemma 9.* *If* $p>1$, $k\neq 1$, $F(t)\geqq 0$, *and*

$$F_1(v)=\int_0^v F(t)dt,\quad (k>1),\quad F_1(v)=\int_v^{\pi}F(t)dt,\quad (k<1),$$

then

$$\int_0^{\pi}t^{-k}F_1^{\,p}\,dt<\left(\frac{p}{|k-1|}\right)^p\int_0^{\pi}t^{-k}(tF)^p\,dt.$$

Lemma 10.† *If* $1<p\leq 2$ *and* $f(x)$ *is integrable*-L^p, *then the series*

$$\sum\frac{|a_n|^p+|b_n|^p}{n^{2-p}}$$

is convergent.

Proof of *Theorem B.* Let $\alpha=1/p$. Since

$$n\bar{c}_n=\frac{2n}{\pi}\int_0^{\pi}\psi(t)\sin nt\,dt=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}\cos nt\,dt$$

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*See Hardy, Littlewood and Pólya (8), p. 245.

†See Hardy and Littlewood (6).

we have

$$-\frac{\pi}{2}\bar{t}_n^{\alpha}=\int_0^{\pi}\psi(t)\frac{d}{dt}L_n^{\alpha}(t)dt$$

$$=\int_0^{\pi/n}\psi(t)\frac{d}{dt}L_n^{\alpha}(t)dt+\int_{\pi/n}^{\pi}\psi(t)\frac{d}{dt}l_n^{\alpha}(t)dt+\int_{\pi/n}^{\pi}\psi(t)\frac{d}{dt}k_n^{\alpha}(t)dt$$

$$=I_1+I_2+I_3, \text{ say.}$$

By *Lemmas 8* and *9* we have

$$\sum_{n=1}^{\infty}\frac{|I_1|^p}{n}\leqq A\sum_{n=1}^{\infty}n^{-1+p}\left(\int_0^{\pi/n}|\psi(t)|dt\right)^p$$

$$\leq A\int_0^{\pi}v^{-1-p}\,dv\left(\int_0^{v}|\psi(t)|\,dt\right)^p$$

$$\leq A\int_0^{\pi}t^{-1-p}(t|\psi(t)|)^p\,dt$$

$$=A\int_0^{\pi}t^{-1}|\psi(t)|^p\,dt<\infty,$$

$$\sum_{n=1}^{\infty}\frac{|I_3|^p}{n}\leq A\sum_{n=1}^{\infty}n^{-1-p}\left(\int_{\pi/n}^{\pi}|\psi(t)|t^{-2}\,dt\right)^p$$

$$\leq A\int_0^{\pi}v^{-1+p}\,dv\left(\int_v^{\pi}|\psi(t)|t^{-2}dt\right)^p$$

$$\leq A\int_0^{\pi}t^{-1+p}(t^{-1}|\psi(t)|)^p\,dt$$

$$=A\int_0^{\pi}t^{-1}|\psi(t)|^p\,dt<\infty.$$

Since, by Minkowski's inequality,

$$\left(\sum_{n=1}^{\infty}\frac{|\bar{t}_n^{\alpha}|^p}{n}\right)^{1/p}\leq A\sum_{j=1}^{3}\left(\sum_{n=1}^{\infty}\frac{|I_j|^p}{n}\right)^{1/p}$$

the theorem is proved if we can show that

$$\sum_{n=1}^{\infty} \frac{|I_2|^p}{n} < \infty.$$

Now

$$I_2 = -\frac{n+\frac{\alpha}{2}}{A_n^{\alpha}} \int_{\pi/n}^{\pi} \psi(t) \frac{\sin[(n+\frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{(2\sin\frac{t}{2})^{\alpha}} dt$$

$$-\frac{\alpha}{A_n^{\alpha}} \int_{\pi/n}^{\pi} \psi(t) \frac{\cos\frac{t}{2}\cos[(n+\frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{(2\sin\frac{t}{2})^{\alpha}} dt$$

$$= -\frac{n+\frac{\alpha}{2}}{A_n^{\alpha}} \int_{0}^{\pi} \psi(t) \frac{\sin[(n+\frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{(2\sin\frac{t}{2})^{\alpha}} dt$$

$$+\frac{n+\frac{\alpha}{2}}{A_n^{\alpha}} \int_{0}^{\pi/n} \psi(t) \frac{\sin[(n+\frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{(2\sin\frac{t}{2})^{\alpha}} dt$$

$$-\frac{\alpha}{A_n^{\alpha}} \int_{\pi/n}^{\pi} \psi(t) \frac{\cos\frac{t}{2}\cos[(n+\frac{\alpha}{2})t - \frac{\alpha\pi}{2}]}{(2\sin\frac{t}{2})^{\alpha+1}} dt$$

$$= M_1 + M_2 + M_3, \text{ say.}$$

By *Lemmas* 8 and 9 we have

$$\sum_{n=1}^{\infty} \frac{|M_2|^p}{n} \leq A \sum_{n=1}^{\infty} n^{-1-(\alpha-1)p} \left(\int_0^{\pi/n} t^{-\alpha}|\psi(t)|dt\right)^p$$

$$\leq A\int_0^{\pi} v^{-1+(\alpha-1)p}\, dv \left(\int_0^{v} t^{-\alpha}|\psi(t)|dt\right)^p$$

$$\leq A\int_0^{\pi} t^{-1+(\alpha-1)p} \left(t^{-\alpha+1}|\psi(t)|\right)^p dt$$

$$= A\int_0^{\pi} t^{-1}|\psi(t)|^p dt < \infty, \tag{15}$$

$$\sum_{n=1}^{\infty}\frac{|M_3|^p}{n}\leq A\sum_{n=1}^{\infty}n^{-1-\alpha p}\left(\int_{\pi/n}^{\pi}t^{-\alpha-1}|\psi(t)|dt\right)^p$$

$$\leq A\int_0^{\pi}v^{-1+\alpha p}dv\left(\int_v^{\pi}t^{-\alpha-1}|\psi(t)|dt\right)^p$$

$$\leq A\int_0^{\pi}t^{-1+\alpha p}(t^{-\alpha}|\psi(t)|)^p\,dt$$

$$= A\int_0^{\pi}t^{-1}|\psi(t)|dt < \infty. \tag{16}$$

Now

$$|M_1|\leq A\,n^{-(\alpha-1)}\max\left|\int_0^{\pi}h(t)\,{\cos \atop \sin}\,nt\,dt\right|$$

where

$$h(t) = \psi(t)\,{\sin \atop \cos}\left(\frac{\alpha}{2}t-\frac{\alpha\pi}{2}\right)/\left(2\sin\frac{t}{2}\right)^{\alpha}.$$

Let

$${A_n^* \atop B_n^*} = \frac{2}{\pi}\int_0^{\pi}h(t)\,{\cos \atop \sin}\,nt\,dt.$$

Then A_n^* and B_n^* are Fourier constants of functions which are integrable-L^p. By *Lemma 10*, we have

$$\sum_{n=1}^{\infty}\frac{|A_n^*|^p+|B_n^*|^p}{n^{2-p}}<\infty.$$

Hence

$$\sum_{n=1}^{\infty}\frac{|M_1|^p}{n}\leq A\sum_{n=1}^{\infty}\frac{|A_n^*|^p+|B_n^*|^p}{n^{1+(\alpha-1)p}}<\infty, \tag{17}$$

since $\alpha p = 1$.

It follows from Minkowski's inequality and (15), (16), (17) that

$$\sum_{n=1}^{\infty}\frac{|I_2|^p}{n}<\infty.$$

This completes the proof of the theorem.

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