Covering Graphs

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ABSTRACT

N. Biggs [1] proved that if G is a t-transitive graph, then there exists a covering graph which is also t-transitive. In this paper, we extend this result to local t-transitivity and to edge-transitivity.

Let G be a graph, the group of graph-automorphism of G will be denoted by aut(G). V(G) and E(G) denote the sets of vertices and edges of G, respectively. In this paper, we assume that G is simple, that is, G has no multiple edges and no loops.

A graph G is called *edge-transitive* if for any two distinct edges $\{u_1, u_2\}$, $\{v_1, v_2\} \in E(G)$, there exists $\alpha \in$ aut(G) such that $\alpha\{u_1, u_2\} = \{v_1, v_2\}$, that is, $\alpha(u_1) = v_1$, $\alpha(u_2) = v_2$ or $\alpha(u_1) = v_2$, $\alpha(u_2) = v_1$. If $t \geqslant 0$ is an integer, we define a *t-arc* of G to be a map v: $\{0, 1, \ldots, t\} \rightarrow V(G)$ such that v(i) is adjacent to v(i+1), $v(i) \neq v(i+1)$, for $0 \leqslant i < t$ and $v(i) \neq v(i+2)$ for $0 \leqslant i < t-1$. For simplicity, we write $v_i = v(i)$ and this t-arc by $[v] = (v_0, v_1, \ldots v_t)$. If for any two t-arcs $[v] = (v_0, v_1, \ldots, v_t)$ and $[w] = (w_0, w_1 \ldots, w_t)$ of G with $v_0 = w_0$ there exists $\alpha \in$ aut(G) such that $\alpha[v] = [w]$, then G is called *locally t-transitive.* If for any two t-arcs of G there exists an automorphism of G mapping one to the other, then G is called *t-transitive.*

If K is any group, G a graph and S(G) the set of all 1-arcs of G, then a funcition ϕ : $S(G) \rightarrow K$ satisfying $\phi(v_1, v_2) = (\phi(v_2, v_1))^{-1}$ for any $(v_1, v_2) \in S(G)$, is called a *K-chain* on G. Now, let ϕ be a K-chain on G. We define a covering graph $\widetilde{G} = \widetilde{G}(K, \phi)$ by:

$$V(\widetilde{G}) = K \times V(G),$$

and $E(\widetilde{G}) = \{ \{(k_1, v_1), (k_2, v_2)\} \mid (v_1, v_2) \in S(G), k_2 = k_1\phi(v_1, v_2) \}$. Moreover, if $\alpha \in$ aut(G), $\hat{\alpha} \in$ aut(K) and if there exists a group homomorphism f: aut(G)$\rightarrow$

aut(K) such that $f(\alpha) = \hat{\alpha}$, then we define the *split extension* [K] aut(G) of K by aut(G) as:

$$[K]\ \mathrm{aut}(G) = K \times \mathrm{aut}(G),$$

師大學報 第二十三期

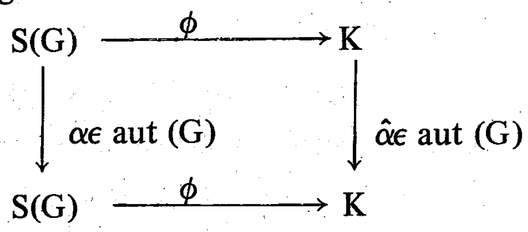
with an operaton defined by:

$(k_1, \alpha_1)(k_2, \alpha_2) = (k_1\ \hat{\alpha}_1\ (k_2), \alpha_1\alpha_2)$, where $\hat{\alpha}_1 \epsilon$ ant(K), $\alpha_1, \alpha_2 \epsilon$ aut(G), $k_1, k_2 \epsilon K$. [K] aut(G) is then a group.

Let ϕ be a K-chain on a graph G, we define

$$\alpha(v_1, v_2) = (\alpha(v_1), \alpha(v_2)), \text{ for all } \alpha \epsilon \mathrm{aut}(G).$$

More, if $\hat{\alpha}\phi = \phi\alpha$ for all $\alpha \epsilon$ aut(G), we say that ϕ is a *compatible K-chain.* Equivalently, the following diagram commutes:



The relation between [K] aut(G) and aut($\tilde{G}$) is:

Lemma If G is a graph, K a group and ϕ a compatible K-chain on G, then [K] aut(G)=aut($\tilde{G}$).

Proof. If $\alpha \epsilon$ aut(G), $k \epsilon K$, then $(k, \alpha) \epsilon$ [K] aut(G). We shall show that (k, α) is an automorphism on $\tilde{G}$. For all $(k', v) \epsilon V(\tilde{G})$, we define

$$(k, \alpha)(k', v) = (k\ \hat{\alpha}\ (k'), \alpha(v)).$$

Then if $\left\{ (k_1', v_1), (k_2', v_2) \right\} \epsilon E(G)$, we have $(v_1, v_2) \epsilon S(G)$ and $k_2' = k_1'\phi(v_1, v_2)$, hence $(k, \alpha)(k_1', v_1) = (k\hat{\alpha}(k_1'), \alpha(v_1))$ and $(k, \alpha)(k_2', v_2) = (k\hat{\alpha}(k_2'), \alpha(v_2))$.

Thus we have

(1) $(\alpha(v_1), \alpha(v_2)) = \alpha(v_1, v_2) \epsilon S(G)$,

(2) $k\ \hat{\alpha}\ (k_2') = k\hat{\alpha}(k_1'\phi(v_1, v_2))$

$= k\hat{\alpha}(k_1')\hat{\alpha}(\phi(v_1, v_2))$

$= k\hat{\alpha}(k_1')\phi(\alpha(v_1, v_2))$

$$= k\hat{\alpha}(k_1')\phi\,(\alpha(v_1), \alpha(v_2)),$$

hence $\left\{ (k, \alpha)\,(k_1', v_1), ((k, \alpha)\,(k_2', v_2) \right\} \epsilon\, E(G)$. It follows that $(k, \alpha) \epsilon$ aut $(\tilde{G})$ and $[K]$ aut $(G) = \text{aut}(\tilde{G})$. Q.E.D.

Let G be a graph. We define K to be the free Z_2-module on E(G) and we define $\phi : S(G) \to K$ by:

$$\phi(v_1, v_2) = \left\{ v_1, v_2 \right\}$$

Then $\phi\,(v_1, v_2) = \phi\,(v_2, v_1) = \left\{ v_1, v_2 \right\}$. Thus $\phi\,(v_1, v_2)\phi(v_2, v_1) = (\left\{ v_1, v_2 \right\})^2 = 1$, hence $\phi(v_1, v_2) = (\phi(v_2, v_1))^{-1}$. This implies that ϕ is a K-chain on G. Since $\phi\alpha(v_1, v_2) = \phi\,(\alpha(v_1), \alpha(v_2)) = \left\{ \alpha(v_1), \alpha(v_2) \right\}$ and $\hat{\alpha}\,\phi\,(v_1, v_2) = \hat{\alpha}\left\{ v_1, v_2 \right\} = \left\{ \hat{\alpha}(v_1), \hat{\alpha}(v_2) \right\} = \left\{ \alpha(v_1), \alpha(v_2) \right\}$, $\phi\alpha = \hat{\alpha}\phi$. Thus ϕ is a compatible K-chain on G. From these statements, we get the main theorem:

Theorem Let G be a connected graph.

(1) If G is t-transitive, then there exists a covering graph $\tilde{G}$ which is also t-transitive.

(2) If G is locally t-transitive, then there exists a covering graph $\tilde{G}$ which is also locally t-transitive.

(3) If G is edge-transitive, then there exists a covering graph $\tilde{G}$ which is also edge-transitive.

Proof. We define K and ϕ as above, so that ϕ is a compatible K-chain on G. We let $\tilde{G} = \tilde{G}(K, \phi)$.

(1) If G is t-transitive, let $((k_0, v_0), \ldots, (k_t, v_t))$ and $((k_0', v_0'), \ldots, (k_t', v_t'))$ and $((k_0, v_0), \ldots, (k_t, v_t))$ be two t-arcs in $\tilde{G}$. Then $(v_0, \ldots, v_t)$ and $(v_0', \ldots, v_t')$) are two -t-arcs in G. Since G is t-transitive, there exists $\alpha \epsilon$ aut(G) such that $\alpha(v_0, \ldots \ldots, v_t) = (v_0', \ldots, v_t')$. Put $k^* = k_0'(\hat{\alpha}(k_0))^{-1}$. Then $(k^*, \alpha) \epsilon\, [K]$ aut (G). By Lemma, $(k^*, \alpha) \epsilon$ aut$(\tilde{G})$. It is sufficient to show that

$$(k^*, \alpha)\,(k_i, v_i) = (k_i', v_i') \quad \text{for } 1 \leqslant i \leqslant t.$$

We shall prove it by induction.

(i) If i=0, then $(k^*, \alpha)\,(k_0, v_0) = (k^*\hat{\alpha}(k_0), \alpha(v_0)) = (k_0'\,(\hat{\alpha}(k_0))^{-1}\hat{\alpha}(k_0), \alpha(v_0)) = (k_0', v_0')$.

(ii) Assume that i=j−1. Then $(k^*, \alpha)(k_{j-1}, v_{j-1}) = (k'_{j-1}, v'_{j-1})$ and $k'_{j-1} = k^*\hat{\alpha}(k_{j-1})$. Since (k_j, v_j) is adjacent to (k_{j-1}, v_{j-1}) and (k'_j, v'_j) is adjacent to (k'_{j-1}, v'_{j-1}), we get $k_j = k_{j-1}\phi(v_{j-1}, v_j)$ and $k'_j = k'_{j-1}\phi(v'_{j-1}, v'_j)$. Thus

$$k'_j = k'_{j-1}\phi(v'_{j-1}, v'_j) = k^*\hat{\alpha}(k_{j-1})\phi(v'_{j-1}, v'_j)$$
$$= k^*\hat{\alpha}(k_{j-1})\phi(\alpha(v_{j-1}), \alpha(v_j))$$
$$= k^*\hat{\alpha}(k_{j-1})\hat{\alpha}\phi(v_{j-1}, v_j)$$
$$= k^*\hat{\alpha}(k_{j-1}\phi(v_{j-1}, v_j)) = k^*\hat{\alpha}(k_j).$$

Hence $(k^*, \alpha)(k_j, v_j) = (k'_j, v'_j)$. By the principle of induction, we get the required result.

(2) If G is locally t-transitive, let $((k_0, v_0), \ldots, (k_t, v_t))$ and $((k_0, v_0), (k'_1, v'_1), \ldots, (k'_t, v'_t))$ be two t-arcs in $\tilde{G}$. Then $(v_0, v_1, \ldots, v_t)$ and $(v_0, v'_1, \ldots, v'_t)$ are two t-arcs in G. Since G is locally t-transitive, there exists $\alpha \epsilon$ aut (G) such that $\alpha(v_0, v_1, \ldots, v_t) = (v_0, v'_1, \ldots, v'_t)$. Put $k^* = k'_1(\hat{\alpha}(k_1)^{-1})$. Then $(k^*, \alpha) \epsilon [K]$ aut(G) and $(k^*, \alpha) \epsilon \mathrm{aut}(\tilde{G})$. It is sufficient to show that

$$(K^*, \alpha)(k_i, v_i) = (k'_i, v'_i) \text{ for } 1 \leqslant i \leqslant t.$$

This can be proved by the same argument as in (1). Hence $\tilde{G}$ is locally t-transitive.

(3) If G is edge-transitive, let $\{(k_0, v_0), (k_1, v_1)\}$ and $\{(k'_0, v'_0), (k'_1, v'_1)\}$ be two edges in G. Then $\{v_0, v_1\}$ and $\{v'_0, v'_1\}$ are two edges in G. Since G is edge-transitive, there exists $\alpha \epsilon$ aut(G) such that $\alpha\{v_0, v_1\} = \{v'_0, v'_1\}$. There are two cases:

(1) $\alpha(v_0) = v'_0$ and $\alpha(v_1) = v'_1$. In this case, by the same argument as in (1), we get $(k^*, \alpha)(k_1, v_i) = (k'_i, v'_i)$ for i = 0,1.

(ii) $\alpha(v_0) = v'_1$ and $\alpha(v_1) = v'_0$. In this case. let $k^* = k'_0(\hat{\alpha}(k_1))^{-1}$, then $(k^*, \alpha) \epsilon [K]$ aut (G) and $(K^*, \alpha) \epsilon \mathrm{aut}(\tilde{G})$. Thus $(k^*, \alpha)(k_1, v_1) = (k^*\hat{\alpha}(k_1), \alpha(v_1)) = (k'_0(\hat{\alpha}(k_1))^{-1}\hat{\alpha}(k_1), v'_0) = (k'_0, v'_0)$ and $(k^*, \alpha)(k_0, v_0) = (k^*\hat{\alpha}(k_0), \alpha(v_0)) = (k^*\alpha(k_0), v'_1)$. But $k_0 = k_1\phi(v_1, v_0)$ and $k'_1 = k'_0\phi(v'_0, v'_1)$, hence

$$k'_1 = k'_0\phi(v'_0, v'_1) = k^*\hat{\alpha}(k_1)\phi(v'_0, v'_1)$$

$$= k^* \hat{\alpha} (k_1) \phi (\alpha(v_1), \alpha(v_0))$$
$$= k^* \hat{\alpha}(k_1) \hat{\alpha} \phi(v_1, v_0)$$
$$= k^* \hat{\alpha}(k_1 \phi(v_1, v_0))$$
$$= k^* \hat{\alpha}(k_0).$$

It follows that $(k^*, \alpha)(k_0, v_0) = (k_1', v_1')$. By (i) and (ii), G is edge-transitive. Q.E.D.

By Theorem, we see that if there exists at least one t-transitive (locally t-transitive, edge-transitive) graph for any value of t, since there are infinitely many covering graphs, there exist infinitely many t-transitive (locally t-transitive, edge-transitive) graphs.

REFERENCES

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中 文 摘 要

本文中我們首先定義覆蓋圖形，然後利用其性質證明可以找到無限多個線遞移，或 t- 遞移或局部 t- 遞移圖形。