NUCLEAR STRIPPING REACTION
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One of the direct reactions to the nuclear process is called the nuclear stripping, and its inverse is called the pick-up reaction. If a complex bombarding particle encounters a nucleus, it can break up on impact such that only one part of it interacts strongly with the nucleus and the other part leaves with practically no interaction. This was first recognized by Oppenheimer and Phillips[1] in analyzing the low-energy (d,p) reactions[2]. It was observed experimentally that (d, p) reactions were more frequent than (d,n) reactions. This is completely opposite from what would be expected if the reaction had proceeded through the formation of the compound nucleus. Because of the absence of the Comlomb barrier, there would be a preponderance of the (d,n) reactions over the (d,p) reactions.

Oppenheimer and Phillips explained the reaction by stating that the deuteron is a loosely bound system with a small binding energy and a large average separation (~4F). When it approaches the target nucleus the proton is detached from the deuteron due to the Coulomb field, whereas the neutron is captured. At low energies, this is known as the Oppenheimer and Phillips process, although it is now more commonly known as the stripping process, at both low and high energies.[3] In the high-energy region, the (d,p) and (d,n) reactions are equally probable. But what distinguishes the reaction from other nuclear-reaction mechanisms is that the noninteracting component of the bombarding particle travels off predominantly in

the forward direction, i.e., in the direction of the incident beam.

The stripping process at high energies was first considered by Serber[4] who was interested in very high-energy deuteron. The process can be illustrated as X(d,f)Y, where X is the ground state of an initial nucleus of mass A, and Y a final nucleus having mass A + 1. The final particle f is one of the two nucleons from the incident deuteron, the other nucleon being denoted by c (see Figure 1). Serber's observations put emphasis on the total angular distribution

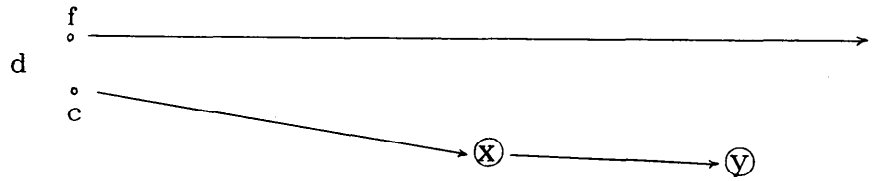

Figure 1

of all particles f irrespective of what states, bound or virtual, of nucleus Y are formed.

The theoretical work of Butler[5] has very successfully demonstrated that the angular distribution of the forward peak is given by the square of the spherical Bessel function of order 1, where 1 is the angular momentum of the state in which c [it is neutron in the case of the (d,p) reaction or proton in the case of the (d,n) reaction] is captured. It is a single-step process, without the formation of a compound nucleus. At short distances, due to the strong interaction between the bombarded nucleus and c, c is captured. The uncaptured nucleon f proceeds in the forward direction giving a forward peak. Eventually, it was found that a large number of nuclear reactions[6], such as $P^{31}(\alpha,p)S^{34}$, $Na^{23}(d,p)Na^{24}$, $Li^{7}(p,d)Li^{6}$, $C^{13}(Hc^{3},\alpha)C^{12}$, and many others showed charac-

teristics of "stripping" or the inverse "pick-up" reaction by the forward peaking. The reactions (p,d) and (n,d) are known as the pick-up reactions. The (p,d) process shows that when the incident proton approaches very closely to the target nucleus, there is a strong interaction between the proton and an outer neutron, resulting in the formation of a deuteron which is then emitted. A similar explanation holds for the (n,d) reaction.

The (α , p) reaction with a forward peak is explained as the stripping of a triton from the alpha particle (see Figure 2), and the (p, α) reaction on the other hand is the reverse reaction, that is, the pick-up of a triton by the proton from the target to form an alpha particle. The reactions such as (α ,p) and (d,p) are known as knock-out reactions, in which the incident particle strikes a particle of another kind and ejected it.

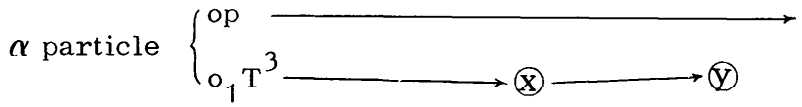

Figure 2

I. Semiclassical Description of the Stripping Reaction

For the sake of simplicity we refer to a (d,p) process in this description, but since we ignore the Coulomb interaction, the arguments are equally valid for a (d,n) process.

Since the number of protons remains unchanged in a reaction, all masses can be written as atomic masses if electron binding-energy differences are ignored. As shown in Figure 3, conservation of energy gives

$$M_d c^2 + T_d + M_x c^2 = M_p c^2 + T_p + M_y c^2 + T_y \qquad (1)$$

where T represents the kinetic energy of each particle in the laboratory system. The $M_y * C^2$ term is used rather than $M_y C^2$ for cases when excited levels occur.

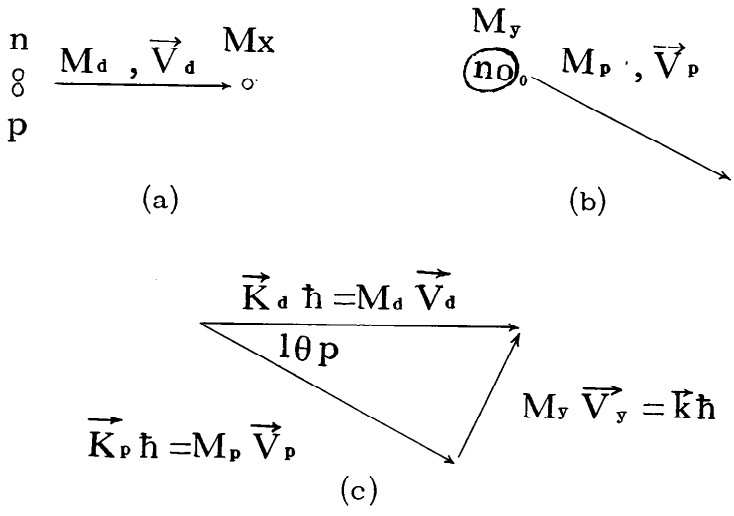

Figure 3

The Q value of the reaction is defined as the difference between the final and initial kinetic energies

$$Q = T_p + T_y - T_d$$
$$= (m_d + M_x) c^2 - (M_y + M_p) c^2 \qquad (2)$$

We use the law of conservation of momentum

$$\vec{P}_d = \vec{P}_y + \vec{P}_p \qquad (3)$$

together with general relativistic relationship between momentum and kinetic energy

$$p^2 = 2MT + \frac{T^2}{C^2} \tag{4}$$

By applying the law of cosines to the momentum triangle, we obtain

$$2M_y T_y + \frac{T_y^2}{C^2} = 2M_d T_d + \frac{T_d^2}{C^2} + 2M_p T_p + T_p^2/C^2 - 2\cos\theta\sqrt{(2M_d T_d + T_d^2/C^2)(2M_p T_p + T_p^2/C^2)} \tag{5}$$

The second-order terms such as $\frac{T_y^2}{C^2}$ are negligible, and expansion of square root gives

$$T_y = \frac{M_d}{M_y} T_d + \frac{M_p}{M_y} T_p - 2\cos\theta \frac{\sqrt{M_d M_p T_d T_p}}{M_y} + \delta\,\mathrm{rel}, \tag{6}$$

where the relativistic correction term is

$$\delta\mathrm{rel} = \frac{1}{2M_y C^2}\left[T_d^2 + T_p^2 - T_y^2 - \cos\theta\sqrt{M_d M_p T_d T_p}\left(\frac{T_d}{M_d} + \frac{T_p}{M_p}\right)\right] \tag{7}$$

In the (d,p) reaction T_y is so small compared with Td and Tp that for the purpose of calculating the relativistic correction, δ rel, we can set Ty = 0, or, we can use the classical expression.

Substitution of Equation (6) into Equation (2) gives the Q-value equation:

$$Q = \left(\frac{M_y + M_p}{M_y}\right) T_p - \left(\frac{M_y - M_d}{M_y}\right) T_d - 2\cos\theta \frac{\sqrt{M_d M_p T_d T_p}}{M_y} + \delta\,\mathrm{rel} \tag{8}$$

In Figure 3, the deuteron, consisting of a neutron and a proton, moves from left toward the target nucleus which is, for the present,

regarded as being very heavy compared with the deutron. The neutron has joined the target nucleus to form a residual nucleus in a specific state. The proton is moving away from the region with a kinetic energy equal to the Q-value for the state in question plus the kinetic energy of the deutron (with $T_y \approx 0$). In the momentum diagram, $\vec{k}_d\hbar$ is the momentum of the deutron, and $\vec{K}_p\hbar$ is the momentum of the proton; hence, $\vec{k}\hbar$ is the momentum imparted to the residual nucleus. When $\vec{K}_d\hbar$, $\vec{K}_p\hbar$ and θ_p are known, $\vec{k}\hbar$ is given by the cosine rule applied to the momentum triangle.

The assumption is now made that the proton does not come within the range of nuclear forces from the target nucleus and is not significantly deflected by the Coulomb forces from the nucleus. The linear momentum $\vec{k}\hbar$ imparted to the nucleus is then the momentum that the neutron bring with it. The orbital angular momentum that the neutron brings with it into the nucleus can also be estimated if a reasonable assumption can be made about the magnitude of the impact parameter of the neutron with respect to the center of mass. If we assume that the impact parameter is approximately equal to R, the nuclear radius, then the orbital angular momentum is $k\hbar R$. This orbital angular momentum is quantized, i.e., it is equal to $\sqrt{l(l+1)}\hbar$ where l is an integer.

This gives

$$KR=\sqrt{l(l+1)} \tag{9}$$

If the above model is not too crude, Equation (9) essentially expresses the fact that it is possible to determine the orbital angular momentum quantum number of the neutron by determining k. As discussed above, k is given when the reaction angle and the kinetic energies of the deuteron and the proton are given. Conclusion

arrived is that when, for stripping to a certain level, only one value of l occurs, the wave number k must satisfy Equation (9), and the geometry of the process is thereby fixed. In other words, the proton must come out at a given reaction angle θ_p which is found by applying the cosine rule to the momentum triangle.

Choosing a radius and applying the cosine rule to the momentum triangle we can make predictions. For a 7-Mev deuteron with $K_d = 0.82F^{-1}$ and a 13-Mev proton with $K_p = 0.79F^{-1}$, we get the following calculated results:

$l = 1, \quad \theta \max = 16^\circ;$
$l = 2, \quad \theta \max = 29^\circ;$
$l = 3, \quad \theta \max = 42^\circ.$

For $l = 0$, the predicted angle for maximum cross section is imaginary ($\cos\theta > 1$), but $\theta = 0$ comes nearest to satisfying the equation. A survey of experimental results shows that the majority of the angular distributions display strong forward maxima, in general agreement with the predictions made above. Specifically, there are many levels with maxima around 17° to 18° and others with maxima around 40°, suggesting stripping with neutrons going into $l = 1$ and $l = 3$ orbits (p- and f- state stripping).

II. Theory of the Stripping Reaction[3]

The stripping reaction can in general be written in the following form:

$$(1 + 2) + 3 \longrightarrow 1 + (2 + 3)$$

The parentheses in the first term denote the bound state of nuclei

1 and 2. Similarly on the right, we have the bound state of nuclei 2 and 3; any one of them may be a single nucleon, proton or neutron. Nucleus 3 is initially free, and nucleus 1 is free in the final state. Thus, in a stripping (d,p) reaction $d + X^A \rightarrow p + X^{A+1}$, we use $1 \equiv p$, $2 \equiv n$, and $3 \equiv X^A$. Similarly, in a pick-up (p,d) reaction $p + X^A \rightarrow d + X^{A-1}$ we have $1 \equiv X^{A-1}$, $2 \equiv n$, $3 \equiv p$. As another example of the pick-up reaction we consider the (p, α) reaction $p + X^A \rightarrow \alpha + X^{A-3}$, in which $1 \equiv X^{A-3}$, $2 \equiv H^3$, and $3 \equiv p$.

As shown, in Figure 4, initially the momentum of particle 3 is $\vec{P}_i$ and that of the bound nucleus (1 + 2) is $-\vec{P}_i$, in the center-of-mass system. Finally, the momentum of the free particle 1 is $\vec{P}_f$ and that of the bound nucleus (2 + 3) is $-\vec{P}_f$.

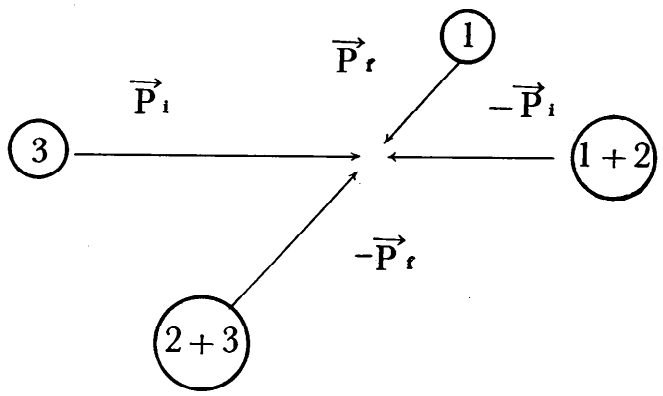

Figure 4

The initial momenta which can be ascribed to 1 and 2 are

$$-\frac{M_1}{M_1+M_2}\vec{P}_i \quad \text{and} \quad -\frac{M_2}{M_1+M_2}\vec{P}_i$$

respectively. Similarly, the final momenta of 2 and 3 are

$$-\frac{M_2}{M_2+M_3}\vec{P}_f \quad \text{and} \quad -\frac{M_3}{M_2+M_3}\vec{P}_f$$

respectively. If we denote the momentum transfer, which is the difference between the final and initial momenta, by $\vec{q}_k$ for the k-th particle where k = 1, 2, 3, we have

$$\vec{q}_1 = \vec{P}_f + \frac{M_1}{M_1+M_2}\vec{P}_i \qquad (1)^*$$

$$\vec{q}_2 = \frac{-M_2}{M_2+M_3}\vec{P}_f + \frac{M_2}{M_1+M_2}\vec{P}_i \qquad (2)^*$$

$$\vec{q}_3 = \frac{-M_3}{M_2+M_3}\vec{P}_f - \vec{P}_i \qquad (3)^*$$

The reduced masses μ_i and μ_f to be associated with the initial and the final momenta are different. μ_i and μ_f are given by

$$\frac{1}{\mu_i} = \frac{1}{M_1+M_2} + \frac{1}{M_3} \qquad (4)^*$$

$$\frac{1}{\mu_f} = \frac{1}{M_1} + \frac{1}{M_2+M_3} \qquad (5)^*$$

The total energy E is given by

$$E = \frac{\vec{P}_i^{\,2}}{2\mu_i} - \varepsilon_{12} = \frac{\vec{P}_f^{\,2}}{2\mu_f} - \varepsilon_{23}, \qquad (6)^*$$

where ε_{12} and ε_{23} are the binding energies of the initially and finally bound systems, (1+2) and (2+3), respectively. The Q-value for the reaction is given by

$$Q = \frac{\vec{P}_f^{\,2}}{2\mu_f} - \frac{\vec{P}_i^{\,2}}{2\mu_i} \qquad (7)^*$$

$$= \varepsilon_{23} - \varepsilon_{12}$$

If the coordinates of particles 1, 2, and 3 are specified by $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$, then the centers of mass of (1+2) and (2+3) are at

$$\vec{R}_{12}=\frac{M_1\vec{r}_1+M_2\vec{r}_2}{M_1+M_2} \quad \text{and} \quad \vec{R}_{23}=\frac{M_2\vec{r}_2+M_3\vec{r}_3}{M_2+M_3} \tag{8}*$$

respectively. Since the center-of-mass remains fixed, the independent sets of coordinates are both the relative coordinates of the particles in the bound system and the difference between the coordinates of the free particle and the center of mass of these particles. Thus,

initially $\vec{r}_{12}=\vec{r}_1-\vec{r}_2, \qquad \vec{r}_i=\vec{r}_3-\vec{R}_{12}$ (9)*

finally $\vec{r}_{23}=\vec{r}_2-\vec{r}_3, \qquad \vec{r}_f=\vec{r}_1-\vec{R}_{23}$ (10)*

are the set of independent coordinates initially and finally, with the condition

$$M_1\vec{r}_1+M_2\vec{r}_2+M_3\vec{r}_3=0 \tag{11}*$$

that is, the center-of-mass of the whole system is at rest. We can determine the momenta $\vec{P}_{12}$ and $\vec{P}_i$ ($\vec{P}_{23}$ and $\vec{P}_f$) which are canonically conjugate to the coordinates $\vec{r}_{12}$ and $\vec{r}_i$ ($\vec{r}_{23}$ and $\vec{r}_f$), respectively, in terms of the momenta $\vec{P}_1$, $\vec{P}_2$, and $\vec{P}_3$ which canonically conjugate to $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$, respectively. After expressing $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$, in terms of $\vec{r}_{12}$ and $\vec{r}_i$, or $\vec{r}_{23}$ and $\vec{r}_f$, we obtain the following expressions by setting $\hbar=1$:

$$\begin{aligned}\vec{p}_{12}&=\frac{\hbar}{i}\frac{\partial}{\partial\vec{r}_{12}}\approx\sum_{k=1,2}\vec{p}_k\frac{\partial\vec{r}_k}{\partial\vec{r}_{12}}\\ \vec{p}_i&=\frac{\hbar}{i}\frac{\partial}{\partial\vec{r}_i}\approx\sum_{k=1,2,3}\vec{p}_k\frac{\partial\vec{r}_k}{\partial\vec{r}_i}\end{aligned} \tag{12}*$$

Following this procedure, we obtain

$$\vec{p}_{12} = \mu_{12}\left(\frac{\vec{p}_1}{M_1} - \frac{\vec{p}_2}{M_2}\right) \qquad (13)^*$$

$$\vec{p}_1 = \mu_1\left(\frac{\vec{p}_1 + \vec{p}_2}{M_1 + M_2} + \frac{\vec{p}_3}{M_3}\right) \qquad (14)^*$$

$$\vec{p}_{23} = \mu_{23}\left(\frac{\vec{p}_2}{M_2} - \frac{\vec{p}_3}{M_3}\right) \qquad (15)^*$$

$$\vec{p}_f = \mu_f\left(\frac{\vec{p}_1}{M_1} - \frac{\vec{p}_2 + \vec{p}_3}{M_2 + M_3}\right) \qquad (16)^*$$

where

$$\frac{1}{\mu_{12}} = \frac{1}{M_1} + \frac{1}{M_2}$$

$$\frac{1}{\mu_{23}} = \frac{1}{M_2} + \frac{1}{M_3}$$

The kinetic energy operator T_c in the center-of-mass system is therefore

$$T_c = \frac{\vec{P}_{12}^2}{2\mu_{12}} + \frac{\vec{P}i^2}{2\mu_1} + \frac{\vec{P}_{23}^2}{2\mu_{23}} + \frac{\vec{P}f^2}{2\mu_f} \qquad (17)^*$$

The total Hamiltonian for the system is given by

$$H = T_c + V_{12} + V_{13} + V_{23} \qquad (18)^*$$

where V is the interaction potential between the particles indicated by subscripts. If E is the total energy of the three particles in

the center-of-mass system and Ψ is the wave function of the system, then

$$H\Psi=E\Psi \tag{19}*$$

is the Shrodinger equation satisfied by Ψ. If ϕ_{12} and ϕ_{23} represent the wave functions of the initial and final bound states (1 + 2) and (2 + 3) respectively, then the Schrodinger equations satisfied by ϕ_{12} and ϕ_{23} are

$$\left(\frac{\vec{p}_{12}^{\,2}}{2\mu_{12}}+V_{12}\right)\phi_{12}(\vec{r}_{12})=-\varepsilon_{12}\phi_{12}(\vec{r}_{12}) \tag{20}*$$

$$\left(\frac{\vec{p}_{23}^{\,2}}{2\mu_{23}}+V_{12}\right)\phi_{23}(\vec{r}_{23})=-\varepsilon_{23}\phi_{23}(\vec{r}_{23}) \tag{21}*$$

We denote by $X_i(\vec{r}_i)$ and $X_f(\vec{r}_f)$ the wave functions of the free particle 3 with respect to the center-of-mass of (1+2) in the initial state, and of 1 with respect to the center-of-mass of (2+3) in the final state, respectively. The wave functions $\Psi(\vec{r}_{12},\vec{r}_i)$ and $\Psi(\vec{r}_{23},\vec{r}_f)$ can be expanded as follows:

$$\Psi(\vec{r}_{12},\vec{r}_i)=\sum_\alpha \phi_\alpha(\vec{r}_{12})X_\alpha(\vec{r}_i) \tag{22}*$$

$$\Psi(\vec{r}_{23},\vec{r}_f)=\sum_\alpha \phi_\alpha(\vec{r}_{23})X_\alpha(\vec{r}_f) \tag{23}*$$

Where α signifies a complete set of states of the bound system and the free particle. From Equations (22)* and (23)* we obtain

$$x_i(\vec{r}_i)=\int\phi^*_{12}(\vec{r}_{12})\Psi(\vec{r}_{12},\vec{r}_i)d^3r_{12} \tag{24}*$$

$$x_f(\vec{r}_f)=\int\phi^*_{23}(\vec{r}_{23})\Psi(\vec{r}_{23},\vec{r}_f)d^3r_{23} \tag{25}*$$

To obtain the differential equation satisfied by $X_f(\vec{r}_f)$, we note that $p_f^2 = -v_f^2$, since $\vec{p}_f$ is the momentum canonically conjugate to the coordinate $\vec{r}f$, and from Equations (6)*, (17)*, (18)*, and (19)*, we have

$$H\Psi(\vec{r}_{23},\vec{r}_f) = \left(\frac{\vec{p}_{23}^{\,2}}{2\mu_{23}} + \frac{\vec{p}_f^{\,2}}{2\mu_f} + V_{12} + V_{13} + V_{23}\right)\Psi(\vec{r}_{23},\vec{r}_f)$$

$$= \left(\frac{p_f^{\,2}}{2\mu_f} - \varepsilon_{23}\right)\Psi(\vec{r}_{23},\vec{r}_f) \qquad (26)^*$$

or $$\frac{1}{2\mu_f}(V_f^{\,2}+p_f^{\,2})\Psi(\vec{r}_{23},\vec{r}_f) = \frac{\vec{p}_{23}^{\,2}}{2\mu_{23}} + V_{12} + V_{13} + V_{23} + \varepsilon_{23})\,x$$

$$\Psi(\vec{r}_{23},\vec{r}_f)$$

Applying the operator $(V_f^{\,2} + P_f^{\,2})$ to $X_f(\vec{r}_f)$ in Equation (25)* and using Equations (20)*, (21)*, and (26)*, we obtain

$$(V_f^{\,2}+p_f^{\,2})x_f(\vec{r}_f) = 2\mu_f \int \phi_{23}^*(\vec{r}_{23})[V_{12}+V_{13}]\Psi(\vec{r}_{23},$$

$$\vec{r}_f)d^3r_{23} \qquad (27)^*$$

The solution for $X_f(\vec{r}_f)$ can be obtained by using the outgoing Green function

$$-\frac{\exp ip_f|\vec{r}_f'-\vec{r}|}{4\pi|\vec{r}_f'-\vec{r}|}$$

Thus

$$x_f(\vec{r}_f) = -\frac{\mu_f}{2\pi}\int \frac{\exp ip_f|\vec{r}_f-\vec{r}|}{|\vec{r}_f-\vec{r}|}\,\phi_{23}^*(\vec{r}_{23})[V_{12}+V_{13}]\Psi d^3r_{23}d$$

$$(28)^*$$

In order that the total wave function Ψ describes the reaction process $(1+2)+3 \rightarrow 1+(2+3)$. The boundary condition imposed

on $X_f(\vec{r}_f)$ is

$$x_f(\vec{r}_f) \xrightarrow[r_f \to \infty]{} f(\theta, \phi)\, e^{i p_f r_f}/r_f \qquad (29)^*$$

where $f(\theta, \phi)$ is the reaction amplitude. Comparing Equations (29)* and (28)* we obtain

$$f(\theta, \phi) = -\frac{\mu_f}{2\pi} \int \phi^*_{23}(\vec{r}_{23}) e^{-i\vec{p}_f \cdot \vec{r}_f} [V_{12} + V_{13}] \Psi d^3 r_{23} d^3 r_f \qquad (30)^*$$

If we accept the Born approximation which ignores the interaction between the incident particle and the target nucleus in expressing Ψ as the product of the incident plane wave and the wave function of the target nucleus, then we can replace Ψ by

$$\phi_{12}(\vec{r}_{12}) \exp(i\vec{p}_1 \cdot \vec{r}_{12} + i\vec{p}_1 \cdot \vec{r}_3) = \phi_{12}(\vec{r}_{12}) \exp(i\vec{p}_1 \cdot \vec{r}_1)$$

The reaction amplitude $f(\theta, \phi)$ is given by

$$f(\theta, \phi) = -\frac{\mu_f}{2\pi} \int \phi^*_{23}(\vec{r}_{23}) e^{-i\vec{p}_f \cdot \vec{r}_f} [V_{12} + V_{13}] e^{i\vec{p}_1 \cdot \vec{r}_1} \phi_{12}(\vec{r}_{12})$$

$$d^3 r_{23} d^3 r_f = -\frac{\mu_f}{2\pi} \int \phi^*_{23}(\vec{r}_{23}) e^{-i(\vec{q}_3 \cdot \vec{r}_{23} - \vec{q}_1 \cdot \vec{r}_{12})}$$

$$[V_{12} + V_{13}] \phi_{12}(\vec{r}_{12}) d^3 r_{12} d^3 r_{23}, \qquad (31)^*$$

where we have

$$-\vec{p}_f \cdot \vec{r}_f + \vec{p}_1 \cdot \vec{r}_1 = \vec{q}_3 \cdot \vec{r}_{23} - \vec{q}_1 \cdot \vec{r}_{12} \text{ and } d^3 r_{23} d^3 r_f = d^3 r_{23} d^3 r_{12}$$

The reaction cross section (0, 0) is given by the outgoing current divided by the incident current density P_1/μ_1, that is,

$$\sigma_{(1+2)+3 \to 1+(2+3)}(\theta, \phi) = \lim_{R \to \infty} \frac{1}{2i\mu_f} \left(x_f^* \frac{\partial x_f}{\partial r_f} - \frac{\partial x_f^*}{\partial r_f} x_f\right)_{r_f = R} \frac{R^2}{p_1/\mu_1} \qquad (32)^*$$

$$= \frac{\mu_1 p_f}{\mu_f p_1} |f(\theta, \phi)|^2$$

$$=\frac{\mu_i\mu_f}{(2\pi)^2}\frac{p_f}{p_i}x\left|\int\phi^*_{23}(\vec{r}_{23})e^{i(\vec{q}_3\cdot\vec{r}_{23}-\vec{q}_1\cdot\vec{r}_{12})}(V_{12}+V_{13})\phi_{12}(\vec{r}_{12})\right.$$

$$\left.d^3r_{12}d^3r_{23}\right|^2$$

Equations (31)* and (32)* can be simplified further by neglecting the interaction V_{13} between particles 1 and 3. This is done because in the reaction $(1+2)+3 \rightarrow 1+(2+3)$ particles 1 and 3 never appear in a bound state. In the stripping reaction $X^A(d,p)X^{A+1}$, $V_{13}=V_p$, A and the interaction V_{12} equals V_{np}. The V_{12} term contributes a nonvanishing result provided the final state contains components of the core, the target nucleus, left in its ground state. On the other hand, the contribution arising from the interaction V_{13} will be nonvanishing if the final state corresponds to excitation of the core. Generally, in a stripping reaction the contribution of the V_{13} interaction is much less than that of V_{12} unless the final state involves almost purely the excitation of the core. Hence, the cross section can be written in the form of

$$\sigma(\theta,\phi)=\frac{\mu_i\mu_f}{(2\pi)^2}\frac{p_f}{p_i}|F(\theta,\phi)|^2$$

where

$$F(\theta,\phi)=\int\phi^*_{23}e^{i\vec{q}_2\cdot\vec{r}_{23}}d^3r_{23}\int e^{-i\vec{q}_1\cdot\vec{r}_{12}}V_{12}(\vec{r}_{12})\phi_{12}(\vec{r}_{12})d^3r_{12} \qquad (33)^*$$

The same cross section can be obtained by using directly the relationship

$$\lambda=\frac{p_i}{\mu_i}(\theta,\phi) \qquad (34)^*$$

From the golden rule number two,[8] the total translation rate is given by

$$\lambda=\frac{2\pi}{\hbar}|H_{fi}|^2\cdot\frac{dN}{dE} \qquad (35)^*$$

The box approximation gives $\frac{dN}{dE}=\frac{VP_f\mu_f}{2\pi^2\hbar}$ with $P_f dP_f=\mu_f dE$. The volume V and the constant $2\pi^2\hbar$ can be adjusted by the normalization constant of the wave function existing in the matrix element $|H_{fi}|^2$. Substituting $\frac{dN}{dE}$ in Equation (35)* and using Equation (34)* we have

$$\sigma(\theta,\phi)=\frac{\mu_i\mu_f}{c}\frac{P_f}{P_i}|F(\theta,\phi)|^2 \qquad (36)^*$$

where C is the constant and $|F(\theta,\phi)|^2$ corresponds to the perturbation matrix elements $|H_{fi}|^2$.

Using the formula for the expansion of a plane wave in terms of spherical harmonics, we can reproduce Butler's theory from Equation (33)* (see Appendix). Although the theory of Butler neglects all Coulomb effects and the reaction effects of the various outgoing waves in the stripping process, a comparison[9] of the reaction $O^{16}(p,d)O^{15}$ with Butler's results shows that the position of the maximum is well reproduced by the theory. On the other hand, at higher angles, the theoretical curve falls much faster than do the experimental points. It appears that Butler's theory is able to explain, at least in a semiqualitative way, the observed angular distribution. The same conclusion can be drawn for the reaction $C^{14}(p,t)C^{12}$.

The plane-wave approximation discussed above ignores two interactions: (1) the long-range repulsive Coulomb force, and (2) the short-range nuclear interaction. Both these interactions give rise to a distortion of the wave function, and in order to obtain detailed agreement between theory and the experimental results for stripping reactions, as well as for pick-up reactions, it is essential to take these into account, especially if the energies of the incident particles are high enough to enable them to penetrate the Coulomb barrier and approach within the region of the nuclear range. Calculation of the

distorted-wave Born approximation has been made by using a computer program written by Tobocman and coworkers[10], and experimental stripping cross sections are found to agree with distorted-wave Born approximation, as pointed out by J. Rapaport, A. Sperduto, and W. W. Buechner in an article to be published.[11]

In general, the model-independent selection rules can also be used to determine the change of angular momentum and parity in a stripping process. The rules are expressed as follows:

$$|(|J_i - \ell| - \frac{1}{2})| \leq J_f \leq J_i + \ell + \frac{1}{2} \qquad (37)*$$

$$\pi_i \pi_f = (-1)^{\ell} \qquad (38)*$$

where J_i is the initial angular momentum, J_f is the final angular momentum, and 1 has been discussed above in Equation (9). If the angular momentum of the target nucleus is 0+, the final state will have angular momentum $J_f = \ell \pm \frac{1}{2}$ and parity $\pi_f = (-1)^{\ell}$.

III. Applications of Deuteron Stripping Reaction

The (d,n) process at high energies was historically used as the first source of neutrons with energies of the order of 100 Mev. When a beam of deuterons of energy much greater than the binding energy of the deuteron impinges on a target of any substance, neutrons which are fairly homogeneous in energy and strongly collimated in the forward direction are stripped off in considerable intensity.[12] At 190 Mev., the neutron yield from a 0.5-in. Be target is about 2% and the mean energy is about 90 Mev. The half-width of the angular

spread is a function of the atomic number of the target. Expressed in radians, it is $(0.155 + 0,0006\quad)$, i.e., about 10^{o} for 190 Mev.

The stripping reaction in some cases can be used to measure the purity of shell-model states. For instance, in the reaction $P^{31}(d,p)P^{32}$, the target nucleus has angular momentum $J_i = \frac{1}{2}^{+}$ (an odd $2S_{1/2}$ proton), and the ground state of P^{32} is expected to be formed by adding a neutron in a $d_{3/2}$ state. The two j-values, $j_p = \frac{1}{2}$ and $jn = 3/2$ can combine to $J_f = 1^{+}$ or 2^{+}. The 1^{+} state happens to be the ground state. If we apply the model-independent rules (37)* and (38)*, we find that the 1^{+} state can be reached by an $1 = 0$ and an $1 = 2$ neutron, whereas the shell-model state $d_{3/2}$, of course, requires that $1 = 2$. The question raised by Bethe and Butler[13] was: Will a stripping experiment yield an angular distribution consistent with $1 = 2$, with $1 = 0$, or with a mixture of these cases? The experiment was performed by Parkinson and coworkers.[14] They found an angular distribution consistent with $1 = 2$ and very little, if any, trace of $1 = 0$ contribution. In the particular example chosen, the test is a very sensitive one because $1 = 0$ stripping gives a cross section that is one order of magnitude larger at the maximum than does $1 = 2$ stripping under similar circumstances. The result is therefore a great tribute to the shell model. Some other examples can be found in different articles.

APPENDIX

The expression for the cross section is

$$\sigma(\theta,\phi) = \frac{\mu_i \mu_f}{(2\pi)^2}\,\frac{P_f}{P_i}\left|\int \phi^{*}_{23}\, e^{i\vec{q}_2\cdot\vec{r}_{23}}\, d^3r_{23} \int e^{-i\vec{q}_1\cdot\vec{r}_{12}}\, v_{12}(\vec{r}_{12})\, \phi_{12}(\vec{r}_{12})\, d^3r_{12}\right|^2 \tag{A-1}$$

We know the bound-state wave function $\phi_{12}(\vec{r}_{12})$ satisfies

$$\left[-\frac{\nabla^2}{2\mu_{12}}+V_{12}(\vec{r}_{12})+\varepsilon_{12}\right]\phi_{12}(\vec{r}_{12})=0 \qquad \text{(A-2)}$$

Also

$$\left(-\frac{\nabla^2}{2\mu_{12}}-\frac{q_1^2}{2\mu_{12}}\right)e^{-\vec{q}_1\cdot\vec{r}_{12}}=0 \qquad \text{(A-3)}$$

Multiplying Equation (A-2) on the left by $e^{-\vec{q}_1\cdot\vec{r}_{12}}$ and Equation (A-3) on the left by $\phi_{12}(\vec{r}_{12})$, and then subtracting and integrating over $\vec{r}_{12}$, we obtain

$$\int e^{-i\vec{q}_1\cdot\vec{r}_{12}}\left[V_{12}(\vec{r}_{12})+\varepsilon_{12}+\frac{q_1^2}{2\mu_{12}}\right]\phi_{12}(\vec{r}_{12})d^3r_{12}$$

$$=\frac{1}{2\mu_{12}}\int\left[e^{-i\vec{q}_1\cdot\vec{r}_{12}}\nabla_{12}^2\phi_{12}(\vec{r}_{12})-\phi_{12}(\vec{r}_{12})\nabla_{12}^2 e^{-i\vec{q}_1\cdot\vec{r}_{12}}\right]d^3r_{12}$$

$$=0 \qquad \text{(A-4)}$$

where we have used the property that surface contributions vanish. Therefore,

$$\int e^{-i\vec{q}_1\cdot\vec{r}_{12}}V_{12}(\vec{r}_{12})\phi_{12}(\vec{r}_{12})\,d^3r_{12}=-\left(\varepsilon_{12}+\frac{q_1^2}{2\mu_{12}}\right)\times \int e^{-i\vec{q}_1\cdot\vec{r}_{12}}\phi_{12}(\vec{r}_{12})d^3r_{12} \qquad \text{(A-5)}$$

From Equations (A-5), (A-1), and (31)* we obtain (after neglecting the interaction V_{13})

$$f(\theta,\phi)=\frac{\mu_f}{2\pi}\left(\varepsilon_{12}+\frac{q_1^2}{2\mu_{12}}\right)g_1(\vec{q}_1)g_3(\vec{q}_3) \qquad \text{(A-6)}$$

Thus

$$\sigma(\theta,\phi)\simeq\frac{\mu_1\mu_f}{(2\pi)^2}\frac{p_f}{p_i}\left(\varepsilon_{12}+\frac{q_1^2}{2\mu_{12}}\right)^2|g_1(\vec{q}_1)g_3(\vec{q}_3)|^2, \qquad \text{(A-7)}$$

where

$$\mathcal{J}_1(\vec{q}_1) = \int e^{-i\vec{q}_1\cdot\vec{r}_{12}}\phi_{12}(\vec{r}_{12})d^3\vec{r}_{12} \qquad \text{(A-8)}$$

$$\mathcal{J}_3(\vec{q}_3) = \int \phi^*_{23}(\vec{r}_{23})e^{i\vec{q}_3\cdot\vec{r}_{23}}d^3r_{23} \qquad \text{(A-9)}$$

If the bound nuclei (1 + 2) and (2 + 3) are in definite orbital-angular momentum states (I_{12},M_{12}) and (I_{23},M_{23}) respectively, we can write

$$\phi_{12}(\vec{r}_{12}) = \phi_{12}(r_{12})\,Y_{\ell_{12}m_{12}}(\hat{r}_{12}) \qquad \text{(A-10)}$$

$$\phi_{23}(\vec{r}_{23}) = \phi_{23}(r_{23})\,Y_{\ell_{23}m_{23}}(\hat{r}_{23}) \qquad \text{(A-11)}$$

where $Y_{lm}(\hat{r})$ are the usual spherical harmonics, $\hat{r}$ is the unit vector specifying the direction with respect to θ and ϕ , and $\phi(r)$ are the radial wave functions. In both Equations (A-10) and (A-11), the spins of the bound states are ignored. Using the formula for the expression of a plane wave in terms of spherical harmonics

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=\ell}^{\ell} i^{\ell}\,j_{\ell}(kr)\,Y_{\ell m}(\hat{r})Y^*_{\ell m}(\hat{k}) \qquad \text{(A-12)}$$

we have

$$\int e^{i\vec{k}\cdot\vec{r}}Y^*_{\ell m}(\hat{r})d\hat{r} = 4\pi i^{\ell}j_{\ell}(kr)\,Y^*_{\ell m}(\hat{k}) \qquad \text{(A-13)}$$

Therefore, from Equations (A-8), (A-9), (A-10) and (A-11) we obtain

$$\mathcal{J}_1(\vec{q}_1) = 4\pi(-i)^{\ell_{12}}Y_{\ell_{12}m_{12}}(\hat{q}_1)\int r_{12}^2\phi_{12}(r_{12})\,x \qquad \text{(A-14)}$$
$$j_{\ell_{12}}(q_1r_{12})dr_{12}$$

$$\psi_3(\vec{q}_3) = 4\pi(+i)^{\ell_{23}} Y^*_{\ell_{23} m_{23}}(\hat{q}_2) \int r_{23}^2 \phi^*_{23}(r_{23}) j_{\ell_{23}} x \quad \text{(A-15)}$$

$$(q_3 r_{23}) d r_{23}$$

Substituting Equations (A-14) and (A-15) in Equations (A-6) and(A-7) and using the usual procedure of evaluating the cross section by averaging over the initial magnetic quantum number M_{12} and summing over the final magnetic quantum number M_{23}, we obtain

$$f(\theta,\phi)_{(1+2)+3 \longrightarrow 1+(2+3)} = 8\pi(i)^{\ell_{23}-\ell_{12}} \mu_f \left(\varepsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right) x$$

$$Y_{\ell_{12} m_{12}}(\hat{q}_1) Y_{\ell_{23} m_{23}}(\hat{q}_3) R_1(q_1) R_3(q_3) \quad \text{(A-16)}$$

and

$$\sigma(\theta,\phi)_{(1+2)+3 \longrightarrow 1+(2+3)} = 4(4\pi)^2 \mu_i \mu_f \frac{p_f}{p_i} \left(\varepsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right)^2 x$$

$$\frac{1}{2\ell_{12}+1} \sum_{M_{12} M_{23}} |Y_{\ell_{12} m_{12}}(\hat{q}_1) p^*_{\ell_{23} m_{23}}(\hat{q}_3)|^2 R_1^2(q_1) R_3^2(q_3)$$

$$= 4\mu_i \mu_f \frac{p_f}{p_i} \left(\varepsilon_{12} + \frac{q_1^2}{2\mu_{12}}\right)^2 (2\ell_{23}+1) R_1^2(q_1) R_3^2(q_3), \quad \text{(A-17)}$$

where

$$R_1(q_1) = \int_0^\infty r_{12}^2 \phi_{12}(r_{12}) j_{\ell_{12}}(q_1 r_{12}) dr_{12} \quad \text{(A-18)}$$

$$R_3(q_3) = \int_0^\infty r_{23}^2 \phi_{23}(r_{23}) j_{\ell_{23}}(q_3 r_{23}) d r_{23} \quad \text{(A-19)}$$

In Equation (A-17) we have used the identity

$$\sum_{m=-\ell}^{\ell} |Y_{\ell m}|^2 = \frac{2\ell+1}{4\pi}$$

Equation (A-17) is the theory of Bulter.

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