ON THE PAPER OF H.W. KIM

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Introduction: Let H be a fixed infinite dimensional Hilbert space over the complex field C, and let $\delta_A : X \to AX-XA$ be an inner derivation on the algebra B(H) of all bounded operators on H. H.W. Kim [4] proved that if a compact operator K lies in the intersection of weak closure of range of δ_A and the commutant of A then K is quasinilpotent. The purpose of this note is via trace argument to reprove his theorem. In what follows, we will denote $R(\delta_A) = \{AX-XA: X \in B(H)\}$ and $R(\delta_A)^{-w}$ will denote the closure of $R(\delta_A)$ in the weak operator topology $\{A\}'$ will denote the commutant of A.

Lemma. Let $A \in B(H)$, and let $\{X_n\}$ be a sequence of operators in B(H) such that $\{AX_n - X_nA\}$ converges weakly to a finite range operator C which is in the commutant of A, then A is nilpotent.

Proof: The mapping $X \to \text{trace}(CX)$ is a continuous (hence weakly continuous) linear functional on B(H) and annihilates $R(\delta_A)$. Therefore for any positive integer $k \geq 2$ trace $(C^k)=0$. Let $R=R(C) + R(C^*)$. Then R is closed because it is the sum of two finite-dimensional subspaces, also R reduces C and $C|R^{\perp}=0$. Hence if B is the restriction of C to R, trace(B^k)=trace $(C^k)=0$ for $k \geq 2$. Since B is an operator on a finite dimensional space. Jacobson lemma[3] implies that B is nilpotent, therefore C is also nilpotent.

Theorem A: *If K is a compact operator in $R(\delta_A)^{-w} \cap \{A\}'$ then K is quasinilpotent.*

Proof: Suppose K is not quasi-nilpotent and let $\lambda \in \sigma(K)$ with $\lambda \neq 0$. Since K is compact it is possible to find a circle Γ around λ such that $\sigma(K) \setminus \{\lambda\}$ is contained in the exterior of Γ. Let

$$E = \frac{1}{2\pi i} \int_{\Gamma} (z-K)^{-1}\, dz;$$

then E is an idempotent in $\{K\}''$ and R(E) is finite dimemsional ([1], p.579). Hence $E\in\{A\}'$ and from this it follows that KE $\in R(\delta_A)^{-w}\cap\{A\}'$ But KE has finite rank so by Lemma, σ (KE) = (O). And this contradicts the fact ([1], p.569) that $\lambda\in\sigma$ (KE). Hence σ (K)=(O) and the proof is complete.

Remark: The above proof can be modified to show that if $T \in R(\delta_A)^{-w}\cap\{A\}'$ then T cannot have any isolated eigenvalues of finite multiplicity. Combining this with a theorem of Putnam[5] gives that $\partial\sigma(T)\setminus\{O\}\subset$ the left essential spectrum.

Combining the method used in the proof of the theorem with the techniques of Herrero and Salinas [2] we also obtain the following theorem.

Theorem B: If $T\in B(H)$ and $\varepsilon>0$ then there is an operator C such that:

(i) R(C) is one dimensional,

(ii) $\|(C)\|<\varepsilon$, and (iii) $T+C\notin\cup\{R(\delta_A)^{-w}\cap\{A\}' : A\in B(H)\}$

As a consequence we have that

$$B(H)\setminus\bigcup_{A\in B(H)} R(\delta_A)^{-w}\cap\{A\}'$$

is dense in B(H).

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